**Math Academy**

**August 2019**

The topic of the August Math Academy 2019 will be around the topics of *counting in number theory*. This is a very broad description, but the problems below give an idea of some of the questions that appear in this area.

Questions:

Answer the following questions and explain your procedure the best you can. If you could not produce a complete solution but have progress, please explain it as well. It is very important that you try to solve these questions without the aid of calculators or computers, at least for a time, so that you get a grasp of where the difficulty lies in these problems.

You could do these problems with a program of a computer or only using a calculator, and this approach is not “wrong”. Technology, specially the power of machines to do hard computations for us with greater accuracy and smaller error margin is fundamental for today’s world, so we do not discourage you to try this if nothing else comes to your mind after some days of serious thinking. But before throwing yourself to this situation think about the problem! What would you do if you had nothing but your mind, pencil and did not want to compute a ton of operations by hand? Yet, do not fear computations and tables, they are very important too. Look for balance between computation, tables, and algorithms and simplification and formulas.

These questions will be evaluated on more than just correct answers. Some of them might be hard enough that you will not solve them entirely, but to explain what did you try to solve them, how far did you progress, what did you read and what did you miss is very important as well, and will be taken a lot into account.

Finally, also mind the presentation. The reader must be able to understand what you are doing, mathematics should be written as any other text. In our case, in correct English with its regular rules. To convey ideas is very hard many times, if we let the disorder of our presentation to get into the way we make it harder for people to understand us. If you do not have ideas on how to write something look online for texts of mathematics to see them, even if you do not understand the mathematics in it. Relay on graphs, tables, drawings if you need to. Be creative! Do your best.

**Question 1:**

There are 25 prime numbers between 1 and 100. Is it possible to fill a 5 x 5 grid, in such a way that the sum of each column and the sum of each row is the same?

**Question 2:**

A *partition* of a positive integer is a way of writing it as a sum of other positive integers, disregarding order. For example, all partitions of n = 5 are:

5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 +1 +1

We denote the number of partitions of an integer by P(n). For the integer n = 10 answer the following questions:

(1) How many partitions in which all parts are different does it have?
(2) How many partitions in which all parts are odd does it have?
(3) How many partitions in which all parts are even does it have?
(4) How many partitions in which no part is a square does it have?
(5) How many in which all parts are prime numbers?
(6) How many partitions does it have?

The aim of this problem is for you to try to find a way of solving this without having to write all the partitions, which might be too many and is prone to error when trying to find them all. Suppose you live in the 1750's and you are exploring these questions and you need to count in a way that you are sure your counting is mistake proof or at least close to it. Now repeat the same questions with n = 15.

**Question 3:**

(1) We will say that a positive integer is a *squarefree* integer if it is not divisible by any square greater than 1. For example: 7, 10, 30 or 330 are squarefree, but 4, 50 or 72 are not square free. How many squarefree integers are there between 1 and 1000?
(2) How many prime numbers are between 1 and 1000?
(3) The previous two problems are similar, they both ask you to count the amount of numbers with some property (being squarefree , being prime) within the same range. Explain, from your perspective, which one was easier to solve and why?
(4) Repeat part (1) and (2), with the same process you used to solve them, but now in the range 1 to 10000?
(5) How do they compare in difficulty with their corresponding question in the smaller range?

**Question 4:**

Suppose you paint each positive integer into one of three colors: red, blue and green. For example, the coloring could start as:

**1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, …**

And separate them into three sequences: red, blue and green sequences. For our example, these begin as

**1, 3, 8, 11,…**

**2, 4, 7, 10,…**

**5, 6, 9,…**

(1) Pick any number N. Prove that no matter what coloring we do, for one of the three colors you can find inside of it an arithmetic progression of length N. (Note: if you find this too complicated try first the case for only two colors).
(2) Can you find a specific coloring such that none of the three sequences has an infinitely long arithmetic subsequence.
(3) Research what is Szemeredi's Theorem. Explain why his theorem implies our question (1). What is the difference between this theorem and question (1)?
(4) Research what is the Green-Tao Theorem on arithmetic subsequences of prime numbers and find an arithmetic sequence of prime numbers that has length 10.
(5) Make a small report, but serious one, on the history of the problem of finding primes on Arithmetic Progressions. This is a very interesting question that spans centuries of human endeavor. Engage in it and explain it in your words, even if you do not understand all the terms that might be mentioned. Mind the mathematics! Try to use examples, to grasp the progress as the centuries go by and the math evolves. How far have we advanced as human kind?