Compact p-adic Lie Groups.

In the following we will assume G is a locally analytic manifold over $\mathbb{Q}_p$. Our aim is to prove and discuss the following theorem:

Theorem 27.1: Let G be any p-adic Lie group; then there exist a compact open subgroup $G' \leq G$ and an integral valued p-valuation ω on G' defining the topology of G' such that we have:

i) (G', ω) is saturated;

ii) rank G' = dim G.

We will divide this into several steps:

Step 1: Building up the chart.

Let $d = \dim G$. We denote the identity of G by e and we pick a chart of G around e, call it

$$c = (U, \varphi, \mathbb{Q}_p^d),$$

and wlog put $\varphi(e) = 0$.

Recall that the multiplication map

$$m : G \times G \longrightarrow G$$

is locally analytic and so there exists a neighborhood

$e \in V \subseteq G$ such that

$$m(V \times V) \subseteq \mathcal{U}.$$

Let us restrict the chart around e now and take

$$(V, \varphi|_V, \mathbb{Q}_p^d).$$

We have a map:

$$\varphi(V) \times \varphi(V) \xrightarrow{\varphi^{-1} \times \varphi^{-1}} V \times V \xrightarrow{m} \mathcal{U} \xrightarrow{\varphi} \varphi(\mathcal{U})$$

which a composition of locally analytic maps.

$$\therefore \quad F := \varphi \circ m \circ (\varphi^{-1} \times \varphi^{-1}) : \varphi(V) \times \varphi(V) \longrightarrow \varphi(\mathcal{U})$$

is locally analytic. Notice $\varphi(V) \times \varphi(V) \subseteq \mathbb{Q}_p^{2d}$, $\varphi(\mathcal{U}) \subseteq \mathbb{Q}_p^d$ are open sets.

Step 2: Using the right expansion.

By definition of locally analytic fonction for $F: \varphi(V) \times \varphi(V) \to \varphi(\mathcal{U})$ around 0, we can find $F_1, \dots, F_d$ formal power series in x $\&$ y such that

$$F(x,y) = (F_1(x,y), \dots, F_d(x,y))$$

and

$$F_i(X,Y) = \sum_{\alpha, \beta} C_{i,\alpha,\beta} X^{\alpha} Y^{\beta}$$

where $X = (X_1, \dots, X_d)$ $\&$ $Y = (Y_1, \dots, Y_d)$ and α, β are d-tuples.

Recall that

$$F_i : \varphi(V) \times \varphi(V) \longrightarrow \mathbb{Q}_p$$

is such that

$$F_i \in \mathcal{F}_\varepsilon(\mathbb{Q}_p^{2d}; \mathbb{Q}_p)$$

and the $\varepsilon > 0$ is the same for all $1 \le i \le d$.

We emphasize three properties now of these functions:

- <u>Property 1</u>: Recall that the sets

$$\mathbb{Z}_p^d \supseteq p\mathbb{Z}_p^d \supseteq p^2\mathbb{Z}_p^d \supseteq p^3\mathbb{Z}_p^d \supseteq \dots$$

form a system of compact open sets that are a neighborhood of the identity.
Since $\varphi(V)$ is a neighborhood of e we have that for big enough n,

$$p^n\mathbb{Z}_p^d \subseteq \varphi(V).$$

- <u>Property 2</u>:

Pick any x and y in $\underset{\varphi(V) \subseteq}{\mathbb{Q}_p^d}$ and consider

$$\varphi^{-1}(x), \varphi^{-1}(y) \in V.$$

Then $\varphi^{-1}(x)\varphi^{-1}(y) \in \mathcal{U}$ and so, by definition of F,

$$\boxed{\varphi(\varphi^{-1}(x)\varphi^{-1}(y)) = (F_1(x,y), \dots, F_d(x,y)).}$$

Property 3: Since $F_i \in \mathcal{F}_\varepsilon(\mathbb{Q}_p^{2d}; \mathbb{Q}_p)$ we have that

$$\lim_{|\alpha|+|\beta| \to \infty} \|c_{i,\alpha,\beta}\|_p \, \varepsilon^{|\alpha|+|\beta|} = 0.$$

For big enough n we have $p^{-n} \leq \varepsilon$ and so

$$F_i \in \mathcal{F}_{p^{-n}}(\mathbb{Q}_p^{2d}; \mathbb{Q}_p),$$

so that we have

$$\lim_{|\alpha|+|\beta| \to \infty} \|c_{i,\alpha,\beta}\|_p \, p^{-n(|\alpha|+|\beta|)} = 0,$$

that is,

$$\lim_{|\alpha|+|\beta| \to \infty} p^{-\mathrm{val}(c_{i,\alpha,\beta}) - n(|\alpha|+|\beta|)} = 0.$$

Equivalently, for big enough n,

$$\boxed{\lim_{|\alpha|+|\beta| \to \infty} \left(\mathrm{val}_p(c_{i,\alpha,\beta}) + n(|\alpha|+|\beta|)\right) = \infty.}$$

If we fix an n, then for big enough α, β we have

$$\mathrm{val}_p(c_{i,\alpha,\beta}) + n(|\alpha|+|\beta|) \geq n$$

for every i, but it might fail for some α, β. The point is that we can actually, increasing n further, assure it doesn't.

Lemma: If one increases n, we have for all i, α, β that

$$\mathrm{val}_p(c_{i,\alpha,\beta}) + n(|\alpha|+|\beta|) \geq n.$$

Proof of lemma:

Proving that

$$\mathrm{val}_p(c_{i,\alpha,\beta}) + n(|\alpha|+|\beta|) \geq n \quad \forall \ \alpha,\beta,i$$

its equivalent to

$$\|\mathcal{F}_i\|_{p^{-n}} \leq p^{-n},$$

which is what we will prove. Denote by

$$\varepsilon_0 := \|\mathcal{F}_i\|_{\varepsilon},$$

then

$$\|c_{i,\alpha,\beta}\| \, \varepsilon^{|\alpha|+|\beta|} \leq \varepsilon_0,$$

by definition of norm: and so

$$\|c_{i,\alpha,\beta}\| \leq \varepsilon_0 \, \varepsilon^{-|\alpha|-|\beta|}.$$

~~Now compute;~~ Now pick n such that

$$p^n \geq \max\left(\frac{1}{\varepsilon}, \frac{\varepsilon_0}{\varepsilon^2}\right).$$

Compute then:

$$\begin{aligned}
\|F_i\|_{p^{-n}} &= \max\left(p^{-n}, \max_{|\alpha|,|\beta|}\left(\|c_{\alpha,\beta}\| \, p^{-n(|\alpha|+|\beta|)}\right)\right) \\
&\leq \max\left(p^{-n}, \max_{|\alpha|,|\beta|} \varepsilon_0 \underbrace{\left(\frac{p^{-n}}{\varepsilon}\right)}_{\leq 1}^{|\alpha|+|\beta|}\right) \\
&\leq \max\left(p^{-n}, \varepsilon_0\left(\frac{p^{-n}}{\varepsilon}\right)^2\right) \\
&\leq p^{-n}.
\end{aligned}$$

Comment on "proof":

while in class we realized here we put the p^{-n} because each F_i has an expansion

$$F_i(X_i, Y_i) = X_i + Y_i + \sum_{\text{bigger order terms}}$$

which separates the max among those of higher terms & those with $(1,0)$ & $(0,1)$ whose coefficient is 1.

Concretely, for big enough n, we have for all i, α, β:

$$\boxed{\mathrm{Val}_p(c_{i,\alpha,\beta}) + n(|\alpha| + |\beta|) \geq n.} \qquad (*)$$

Step 3: The inverse.

We have done the previous steps for the multiplication map but we can do them as well for the inverse map, which is also locally analytic.

Step 4: Building subgroups.

Because $p^n \mathbb{Z}_p^d \subseteq \varphi(V)$ we know that

$$\varphi^{-1}(p^n \mathbb{Z}_p^d) \subseteq V \subseteq G$$

are compact open subsets of G. Notice that if $x, y \in p^n \mathbb{Z}_p^d$ then

$$\varphi^{-1}(x)\varphi^{-1}(y) = \varphi^{-1}(F_1(x,y), \dots, F_d(x,y))$$

and precisely the fact that $(*)$ happens implies

$$F_i(x,y) \in p^n \mathbb{Z}_p^d$$

So that

$$\varphi^{-1}(x)\varphi^{-1}(y) \in \varphi^{-1}(p^n \mathbb{Z}_p^d)$$

for large enough n. Repeating this with the inverse we conclude that:

$\boxed{\text{For large enough } n,\ \varphi^{-1}(p^n \mathbb{Z}_p^d) \text{ are subgroups open compact of } G.}$

Step 5: Reescaling the power series.

Pick now any of these large n, so that $\varphi^{-1}(p^n \mathbb{Z}_p)$ is a compact open subgroup of G, and define

$$\psi := p^{-n} \varphi.$$

The multiplication map in this chart is given, in coordinates as:

$$\sum_{\alpha, \beta} (p^{-n} c_{i\alpha\beta}) x^\alpha y^\beta,$$

in the ith-entry.

We proved in the lemma that

$$\| F_i \|_{p^{-n}} \leq p^{-n}.$$

That is: for all α, β,

$$\| c_{i,\alpha,\beta} \|_p \, p^{-n(|\alpha| + |\beta|)} \leq p^{-n}$$

That is:

$$\| p^n \|_p \| c_{i,\alpha,\beta} \cdot p^{-n} \|_p \; p^{-n(|\alpha| + |\beta|)} \leq p^{-n}$$

implying

$$\| c_{i,\alpha,\beta} \, p^{-n} \|_p \cdot p^{-n(|\alpha| + |\beta|)} \leq 1$$

$\vdots$

Repeating this by iteration we have that

$$g \longrightarrow g^p,$$
$$(g,h) \longrightarrow [g,h]$$

have convergent power expansions ~~in~~ around e in ψ, converging in all of $\mathbb{Z}_p^d$ with coefficients in $\mathbb{Z}_p$.

We define

$$\boxed{G' := \psi^{-1}(p^2\mathbb{Z}_p^d).}$$