

Thursday August 1, 2019.
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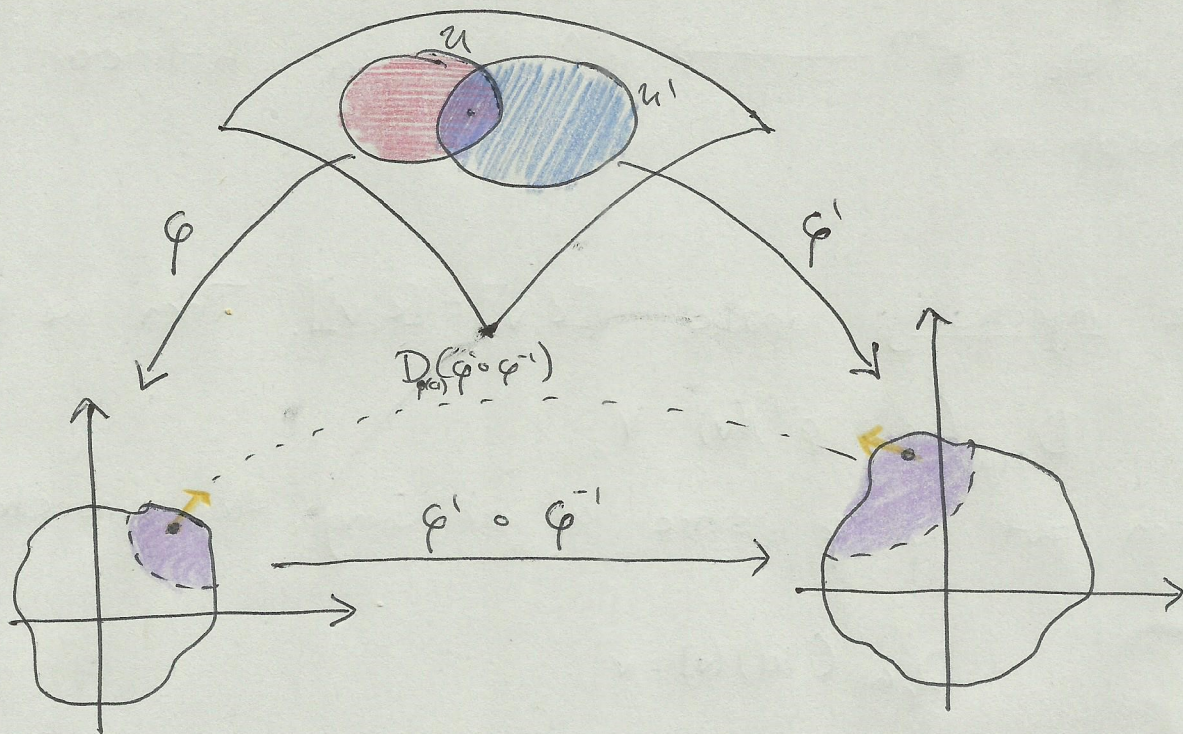
The tangent plane.

Let M be a manifold. We consider pairs (C, v) where $c = (U, \varphi, \mathbb{K}^m)$ is a chart of M and $v \in \mathbb{K}^m$. Fix $a \in M$.

Definition: Two pairs (C, v) and (C', v') are **equivalent** if

$$D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v', \quad \text{~~if this is~~}$$

Recall that different charts in M are equivalent since we are fixing a maximal atlas, so we have:



Recall we proved that $\varphi' \circ \varphi^{-1}$ is strictly diff. since it is locally analytic and that compatible charts have the same dimension. Hence $D_{\varphi(a)}(\varphi' \circ \varphi^{-1}) : \mathbb{K}^m \longrightarrow \mathbb{K}^m$ indeed exists and the condition we are asking makes sense.

This is clearly an equivalence relation and hence we define:

Definition: A tangent vector of M at $a \in M$ is an equivalence class $[c, v]$ of pairs (c, v) as above. The set of all tangent vectors is called the tangent plane at a .

Proposition: Let $c = (U, \varphi, k^m)$ and $c' = (U', \varphi', k^n)$ be two charts for M around a . Then

i) The map

$$\begin{array}{ccc} \theta_c: k^m & \longrightarrow & T_a(M) \\ v & \longmapsto & [c, v] \end{array}$$

is bijective.

ii) $\theta_{c'}^{-1} \circ \theta_c: k^m \longrightarrow k^n$ is a k -linear isomorphism.

Proof:

i) To prove injectivity; suppose $[c, v] = [c, v']$. Then we have

$$D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v',$$

but we can put $\varphi' = \varphi$ since we are using the same chart c .

Hence:

$$D_{\varphi(a)}(\text{id})(v) = v',$$

that is, $v = v'$.

To prove surjectivity: Suppose $[c', v']$ is a Tangent vector. We need to find v such that $[c, v] = [c', v']$, that is,

$$D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v'$$

But $\varphi' \circ \varphi^{-1}$ is an homeomorphism, hence its derivative is too, here we can pick

(2)

$$v := D_{\varphi(a)}(\varphi' \circ \varphi^{-1})^{-1}(v')$$

and this makes the job.

(ii) We need to understand the inverse of Θ_c to get this part. If we have

$$[c, v] = \Theta_c(v) = [c', v'],$$

then we have

$$v = D_{\varphi'(a)}(\varphi \circ (\varphi')^{-1})(v')$$

That is:

$$\Theta_c^{-1}([c', v']) = D_{\varphi'(a)}(\varphi \circ (\varphi')^{-1})(v').$$

Hence:

$$\Theta_c^{-1} \circ \Theta_c(v) = \Theta_c^{-1}([c, v])$$

$$= D_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v)$$

} change the role of the ' (prime) since here we find Θ_c^{-1} , not Θ_c^{-1} .

That is:

$$\Theta_c^{-1} \circ \Theta_c = D_{\varphi(a)}(\varphi' \circ \varphi^{-1})$$

which we know is a k -linear isomorphism (this includes continuity)!

Hence, given this proposition, we can equip $T_a(M)$ with a topological vector space structure by claiming Θ_c is a topological isomorphism, and by (ii) this

is chart independent.

Notice this definition implies that M has dimension n iff $T_a(M)$ has dimension n for all a .

Having defined $T_a(M)$ for each $a \in M$ we define, as a set,

$$T(M) = \coprod_{a \in M} T_a(M).$$

and define

$$\begin{array}{ccc} p_M: T(M) & \longrightarrow & M \\ t & \longmapsto & a \end{array}$$

if $t \in T_a(M)$. We call this the **projection map**.

Pick a chart of M , say $c = (U, \varphi, k^m)$ and consider

$$\begin{array}{ccc} U \times k^m & \xrightarrow{\tau_c} & p_M^{-1}(U) \left(= \coprod_{a \in U} T_a(M) \right) \\ (u, v) & \longmapsto & [v, c] \in T_u(M). \end{array}$$

Notice that

$$\tau_c(u, v) = [v, c] = \theta_c(v), \text{ seen at } u.$$

Hence it is a bijection since the θ_c were. We can then compose:

$$\begin{array}{ccccc} p_M^{-1}(U) & \xrightarrow{\tau_c^{-1}} & U \times k^m & \xrightarrow{\varphi \times \text{id}} & k^m \times k^m = k^{2m} \\ & \searrow \psi_c & & & \end{array}$$

With this we have constructed triples:

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$$(p_M^{-1}(u), \psi_c, k^{2m})$$

as c varies. We want these to be charts for $T(M)$.

Definition: A subset $X \subseteq T(M)$ is called **open** if

$$\tau_c^{-1}(X \cap p_M^{-1}(u))$$

is open in $U \times k^m$ for any chart $c = (U, \varphi, k^m)$ of M .

This definition of openness defines a topology in $T(M)$. We now proceed to prove it is a manifold with the previous charts.

Theorem: i) The map $\tau_c: U \times k^m \longrightarrow p_M^{-1}(u)$ is a homeomorphism with respect to the subspace topology.

ii) p_M is continuous.

iii) $T(M)$ is Hausdorff.

iv) The constructed charts are compatible.

Proof:

Let's pick two charts $c = (U, \varphi, k^m)$ and $c' = (V, \psi, k^m)$ with $U \cap V \neq \emptyset$. We have:

$$\begin{array}{ccccc} & & p_M^{-1}(u) \cap p_M^{-1}(v) & & \\ & & \parallel & & \\ (u \cap v) \times k^m & \xleftarrow{\tau_{c'}} & p_M^{-1}(u \cap v) & \xleftarrow{\tau_c} & (u \cap v) \times k^m \\ & \searrow & \downarrow p_M & \swarrow & \\ & & u \cap v & & \end{array}$$

is:

$$(x, v) \longrightarrow [c', v] \in T_x(M) \longrightarrow (x, D_{\varphi(x)}(\varphi \circ \psi)(v)).$$

If we go further down to coordinates: $\varphi(u|v)$ & $\psi(u|v)$ we get:

$$(x, v) \longmapsto (\varphi \circ \psi^{-1}(x), D_x(\varphi \circ \psi^{-1})(v)).$$

By hypothesis of compatibility the first entry is locally analytic. Now we need to understand the map:

$$(x, v) \longmapsto D_x(\varphi \circ \psi^{-1})(v).$$

This can be further decomposed:

$$\begin{array}{ccccc} (x, v) & \longmapsto & (D_x(\varphi \circ \psi^{-1}), v) & \longmapsto & D_x(\varphi \circ \psi^{-1})(v) \\ \uparrow & & & & \\ \mathbb{R}^{2m} & \longmapsto & \mathcal{L}(K^m, K^m) \times K^m & \longrightarrow & K^m \end{array}$$

In here we use the fact:

Fact [See Prop 6.1]: Suppose $f: U \rightarrow V$ is locally analytic, then f is strictly diff. and $x \mapsto D_x f$ is locally analytic in $C^0(U, \mathcal{L}(K^n; V))$.

Hence the previous map is composition of locally analytic ones. (in particular, continuous and by changing roles, a homeomorphism.)

This actually proves (iv).

But we will use it for (c) as well. Pick $Y \subseteq U \times K^m$ be open. We want to prove $\tau_c(Y)$ is open in $p_M^{-1}(U) \subseteq T(M)$. For that, using the definition, we need

$$\tau_{\tilde{c}}^{-1}(\tau_c(Y) \cap p_M^{-1}(v)) \subseteq V \times K^m$$

is open for any other chart $\tilde{c} = (V, \psi, K^n)$. WLOG we have $V \cap U \neq \emptyset$ so that $n = m$.

We have:

$$\begin{aligned} \tau_c^{-1}(\tau_c(Y) \cap p_M^{-1}(v)) &= \tau_c^{-1}(\tau_c(Y) \cap p_M^{-1}(v) \cap p_M^{-1}(u)) \\ &= \tau_c^{-1}(\tau_c(Y) \cap p_M^{-1}(u \cap v)) \\ &= \tau_c^{-1}(\tau_c(\underbrace{Y \cap ((u \cap v) \times k^m)})) \end{aligned}$$

and this is open since Y is.

This proves (i).

To prove (ii): Notice that $p_M^{-1}(u) = \tau_c(u \times k^m)$ which is open.

To prove (iii): If s, t are such that

$$p_M(s) \neq p_M(t),$$

we find disjoint sets U, V in M with $p_M(s) \in U, p_M(t) \in V$ and so

$$p_M^{-1}(u) \cap p_M^{-1}(v) = \emptyset$$

If $a = p_M(s) = p_M(t)$ and take a chart $\mathcal{U}$ around a .

Since $p_M^{-1}(u) \cong U \times k^m$ and this is Hausdorff separate them there.

$\therefore$ It is Hausdorff.

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Thanks to this proposition we can put a maximal atlas to $T(M)$.

Definition: $T(M)$ equipped with the previous topology and the maximal atlas corresponding to the defined charts is called the **tangent bundle** of M .

Notice that if $\dim M = m$, then $\dim T(M) = 2m$.

Given a map $f: M \longrightarrow N$ between manifolds we know its locally analytic if "locally" it is locally analytic. That is: for all $a \in M$ $\exists$ a chart $U \ni a$ & $V \ni f(a)$ with $f(U) \subseteq V$ such that

$$\psi \circ f \circ \varphi^{-1}: \varphi(U) \longrightarrow \psi(V)$$

i) locally analytic.

Then taking its derivative at the point $x \in \varphi(U)$ we have a map:

$$D_x(\psi \circ f \circ \varphi^{-1}): k^m \longrightarrow k^n$$

and composing with θ_c we get a map

$$\begin{array}{ccc} T_{\varphi^{-1}(x)} M & \xrightarrow{T_{\varphi^{-1}(x)}(f)} & T_{\varphi^{-1}(x)} N \\ \uparrow \theta_c & & \uparrow \theta_{\tilde{c}} \\ k^m & \xrightarrow{D_x(\psi \circ f \circ \varphi^{-1})} & k^n \end{array}$$

This does not depend on the charts c & $\tilde{c}$. We can further define then a map

$$\begin{array}{ccc} T(f): TM & \longrightarrow & TN \\ a & \longmapsto & T_{p_M(a)}(f)(a) \end{array}$$

we have the following

Proposition: Let M be a manifold.

i) The projection map p_M is locally analytic.

ii) For any locally analytic map $f: M \rightarrow N$ the map $Tf: TM \rightarrow TN$ is locally analytic.

iii) We have the chain rule

$$T(f \circ g) = Tf \circ Tg.$$

iv) For any two manifolds M and N we have

$$T(M \times N) \cong TM \times TN$$

as manifolds.

The reason we are interested in TM is to define the following:

Definition: Let M be a manifold and $U \subseteq M$ an open set. A **vector field** ξ of M on U is a locally analytic map

$$\xi: U \longrightarrow T(M)$$

satisfying $p_M \circ \xi = \text{id}_U$. The set of all vector fields on U is denoted by $\Gamma(U, TM)$.

Next class we are going to study the structure of vector fields in detail. For now we are going to move on.

Definition: A **Lie group** G is a manifold which also carries the structure of a group such that the multiplication map m is locally analytic

The basic properties are:

Properties:

i) For any $h \in G$, we have

$$l_h: G \longrightarrow G, \quad r_h: G \longrightarrow G$$

$$g \longmapsto hg, \quad g \longmapsto gh$$

are locally analytic isomorphisms.

ii) For any two elements $g, h \in G$ the map

$$T_g(l_{hg^{-1}}): T_g(G) \longrightarrow T_h(G)$$

is a k -linear isomorphism; in particular

$$T_e(l_g): T_e(G) \longrightarrow T_g(G)$$

is an isomorphism for any $g \in G$.

iii) Any Lie Group is n -dimensional with $n = \dim_e G$.

iv) The diagram

$$\begin{array}{ccc} T_{(g,h)}(G \times G) & \xrightarrow{T(m)} & T_{gh}(G) \\ & \searrow \begin{matrix} T(p_1) \\ \times T(p_2) \end{matrix} & \nearrow T_g(r_h) + T_h(l_g) \\ & T_g(G) \times T_h(G) & \end{array}$$

is commutative. In particular

$$\begin{array}{ccc} T_{(e,e)}(G \times G) & \xrightarrow{T_m} & T_e(G) \\ & \searrow & \nearrow + \\ & T_e(G) \times T_e(G) & \end{array}$$

is commutative.

Proof:

(i), (ii) & (iii) are straight forward.

We do (iv):

Define maps

$$\begin{aligned} i_h: G &\longrightarrow G \times G \\ x &\longmapsto (x, h) \end{aligned}$$

$$\begin{aligned} j_g: G &\longrightarrow G \times G \\ x &\longmapsto (g, x) \end{aligned}$$

These are locally analytic maps which satisfy

$$Pr_1 \circ i_h = id_G, \quad Pr_2 \circ i_h \equiv h$$

$$Pr_2 \circ j_g = id_G, \quad Pr_1 \circ j_g \equiv g.$$

Hence taking tangent map:

$$\left. \begin{aligned} T(Pr_1) \circ T(i_h) &= id_{T(G)} \\ T(Pr_2) \circ T(i_h) &= 0. \end{aligned} \right\} (*)$$

We know that the isomorphism ~~between~~

$$T(G_1 \times G_2) = T(G_1) \times T(G_2)$$

is the one induced by the projections. Hence:

$$\begin{aligned} G &\xrightarrow{i_h} G \times G \\ &\quad \downarrow \text{Tangent} \\ T_g G &\xrightarrow{T(i_h)} T_{(g,h)}(G \times G) \xrightarrow{T(Pr_1) \times T(Pr_2)} T_g(G) \times T_h(G) \end{aligned}$$

is exactly $t \mapsto (t, 0)$ by (*).

In analogous way

$$\begin{array}{ccccc} T_h G & \xrightarrow{T_j g} & T_{(g,h)}(G \times G) & \xrightarrow{T_{p_1} \times T_{p_2}} & T_g G \times T_h G \\ t & \longmapsto & & & (0, t). \end{array}$$

Hence if $(t_1, t_2) \in T_g G \times T_h G$ then:

$$\begin{aligned} (T_{p_1} \times T_{p_2})^{-1}(t_1, t_2) &= \underbrace{(T_{p_1} \times T_{p_2})^{-1}}_Q (Q \circ T_g(\text{in})(t_1) + Q \circ T_h(\text{jg})(t_2)) \\ &= T_g(\text{in})(t_1) + T_h(\text{jg})(t_2). \end{aligned}$$

So we find the explicit form of the inverse.

With this let's compute:

$$\begin{aligned} T_{(g,h)}(m) \circ ((T_{g,h})(p_{r_1}) \times T_{(g,h)})^{-1} \\ = T_{(g,h)}(m) (T_g(\text{in}) + T_h(\text{jg})) \\ = T_g(m \circ \text{in}) + T_h(m \circ \text{jg}) \\ = T_g(r_h) + T_h(l_g). \end{aligned}$$

as desired. In particular, multiplying at identity corresponds to adding.

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Proposition: 

The inverse is locally analytic.

Proof:

Consider the map

$$\begin{array}{ccc} \mu: G \times G & \longrightarrow & G \times G \\ (x, y) & \longmapsto & (xy, y). \end{array}$$

It is a bijection and it is locally analytic. (7)
 Look at the diagram:

$$\begin{array}{ccc}
 T_{(g,n)}(G \times G) & \xrightarrow{T_{(g,n)}(\mu)} & T_{(g,h,n)}(G \times G) \\
 \cong \downarrow & \searrow (T_{(g,n)}(n), \dots) & \cong \downarrow \\
 T_g(G) \times T_n(G) & \xrightarrow{\quad} & T_{g,n}(G) \times T_n(G) \\
 (t_1, t_2) & \xrightarrow{\quad} & (T_g(r_n)(t_1) + T_n(I_g)(t_2), t_2)
 \end{array} \quad (A)$$

This is commutative by (iv). Let's see the lower row and assume (t_1, t_2) is in the kernel. Hence $t_2 = 0$ and so

$$\underbrace{T_g(r_n)}_{\text{This is an isomorphism}}(t_1) = 0$$

so that $t_1 = 0$.

$\therefore$ This lower row is injective, hence so is the map $T_{(g,n)}(\mu)$. Since all ~~exogenous~~ vector spaces are of finite dimension we conclude $T_{(g,n)}(\mu)$ is a bijection.

We conclude by the inverse function theorem that the inverse is a locally analytic map. Now write the inverse as:

$$\begin{array}{ccccc}
 G & \xrightarrow{j_e} & G \times G & \xrightarrow{\mu^{-1}} & G \times G \xrightarrow{pr_1} G \\
 g & \longmapsto & (e, g) & & (g^{-1}, g)
 \end{array}$$

A corollary of this is the structure at the identity:

Corollary:

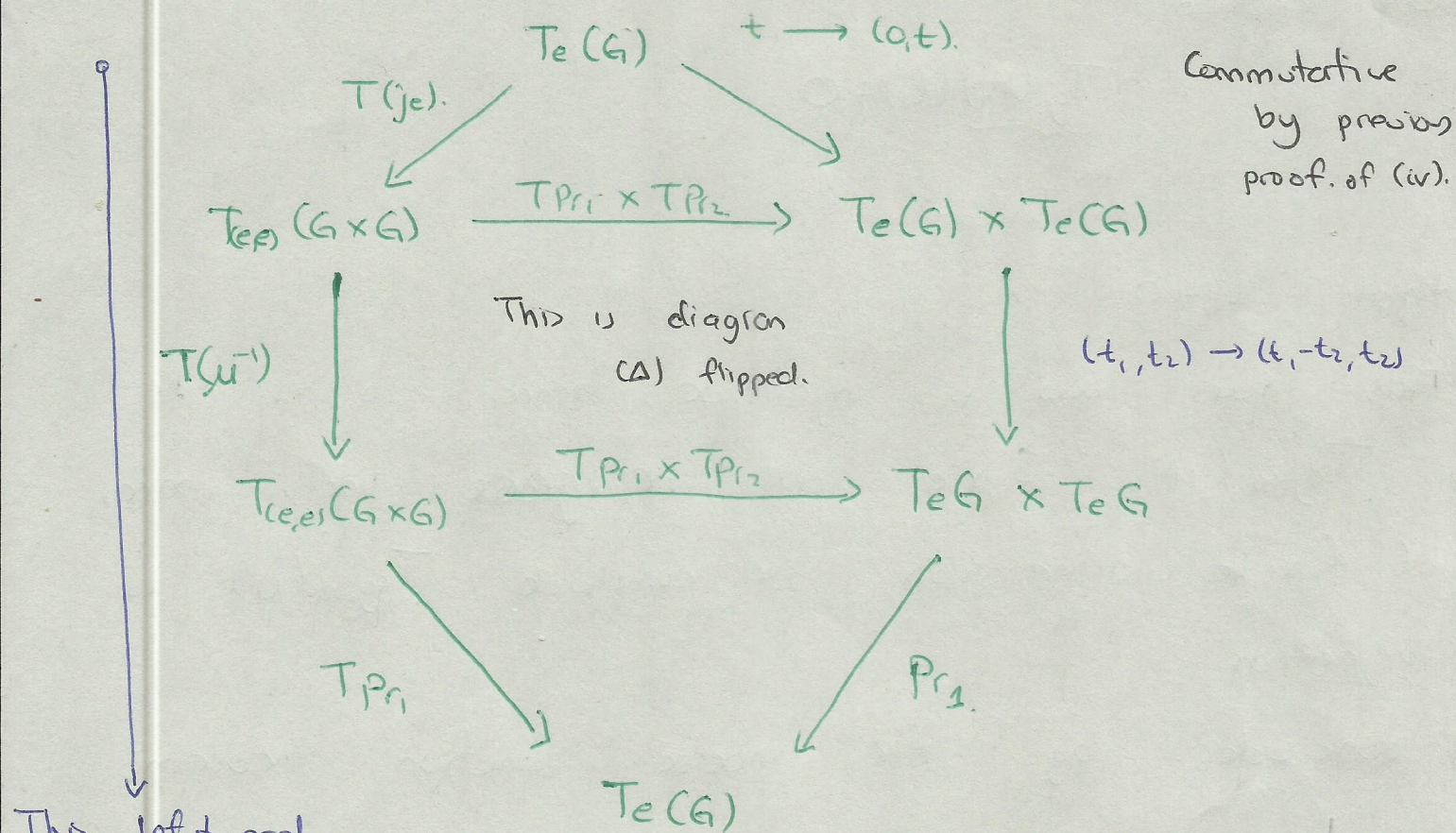
- i) The tangent map $T_e(z)$ coincides with multiplication by -1 .
- ii) For every $n \in \mathbb{Z}$ the map

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^n \end{aligned}$$

has derivative at e equal to multiplication by n .

Proof:

i) Notice that:



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is z . since
 $z = Pr_1 \circ \mu^T \circ j_e$.

Here all is commutative, but left RHS is

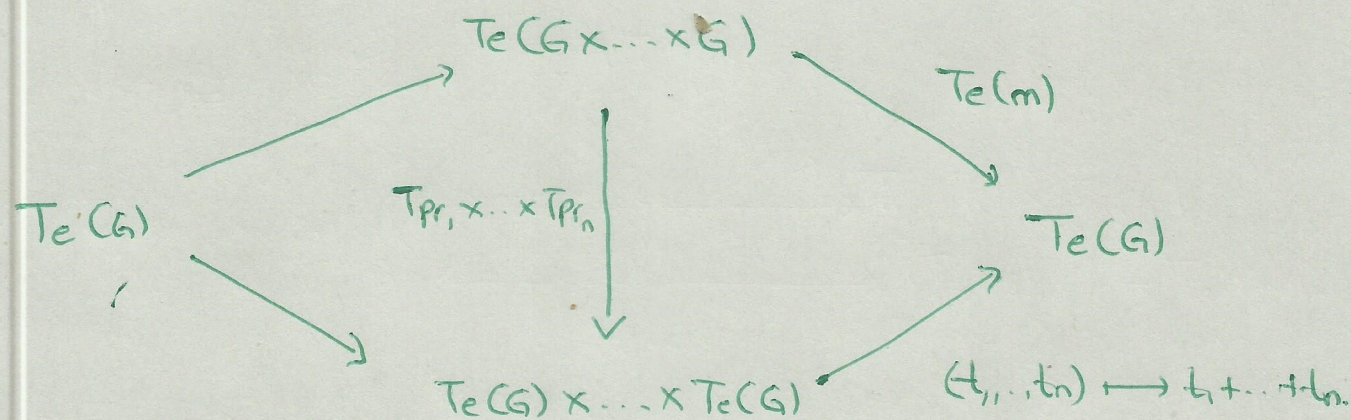
$$t \rightarrow (0, t) \rightarrow (-t, t) \rightarrow -t.$$

(ii) $n=0$ is trivial.

(8)

$n < 0$ follows by (i) and case $n > 0$ by chain rule.

Now notice the diagram:



This is commutative, and gives what we want.