Buildings.

Definition (Building): A building is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ, called apartments satisfying:

(B0) Each apartment Σ is a Coxeter Complex.

(B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

(B2) If Σ & Σ' are two apartments containing A & B, then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing A and B pointwise.

Remark: Suppose A, B are the empty simplex. Then $\Sigma \longrightarrow \Sigma'$, in condition (B2), is just ~~the~~ an isomorphism, from which we deduce all apartments are isomorphic.

As a consequence of this, Δ is finite dimensional, with dimension equal to that of the one of apartments.

It is also a chamber complex since, using (B1), we can join with chambers in a particular apportment any two given simplices

Definition: Let Δ be a building. Any collection $\mathcal{A}$ of subcomplexes of Δ satisfying the axioms will be called a system of apartments.

Proposition: In the presence of (B0) & (B1) the following are equivalent:

(B2): If Σ & Σ' are two apartments containing A & B, then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing A & B pointwise.

(B2'): Let Σ & Σ' be apartments containing A, C, with C a chamber of Σ. Then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing A & C pointwise.

(B2''): Let Σ and Σ' be two apartments containing a simplex C that is a chamber of Σ. Then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing every simplex in $\Sigma \cap \Sigma'$.

Proof:

(B2) $\Rightarrow$ (B2'): This is obvious.

(B2') $\Rightarrow$ (B2): Take A & B any two simplices contained in Σ & Σ'.

Pick a chamber $C \in \Sigma$ with $A \leq C$, and pick one $D \in \Sigma'$ with $B \leq D$. Using (B1) there is ~~an~~ ~~is an~~ a chamber Σ'' containing D & C.

Using (B2') we can find isomorphism:

$$\Sigma \longrightarrow \Sigma'' \quad \text{fixing} \quad C \text{ \& } B.$$

$$\Sigma'' \longrightarrow \Sigma' \quad \text{fixing} \quad D \text{ \& } A.$$

$\therefore$ $\Sigma \longrightarrow \Sigma'$ is an isomorphism fixing A & B, since in one it is fixed & in the other one a chamber that contains them is fixed.

This proves (B2).

(B2") $\Rightarrow$ (B2'): This is obvious.

(B2') $\Rightarrow$ (B2"):

Proposition: Δ is colorable. Moreover, the map in (B2) can be chosen to be type preserving.

Proof:

Fix a chamber C and assign types to its vertices. Pick any apartment $\Sigma \supseteq C$, and because it is ~~not~~ a Coxeter complex it itself is colorable so the type assignement extends to ~~all~~ Σ.

If we pick another Σ' we have that, in $\Sigma \cap \Sigma'$ the colorations coincide and this follows since, ~~as pass~~

$$\tau_{\Sigma'} = \tau_{\Sigma} \circ \phi,$$

where $\phi : \Sigma' \longrightarrow \Sigma$ is the isomorphism fixing $\Sigma \cap \Sigma'$ (and hence C).

Hence as we vary Σ in the apartments containing C we get a well defined type function. Applying (B1) to C & a vertex, we find we have colored all of Δ.

For the second part, consider (B2') instead of (B2). Since it fixes C pointwise it fixes ~~the~~ the coloration.

$-\circ-$

Now we prove:

Proposition: Let Σ be a coxeter complex associated to (W,S). Given $A \in \Sigma$ with cotype $J := S \setminus \tau(A)$, we have $lk_{\Sigma}(A)$ is isomorphic to $\Sigma(W_J, J)$. In particular, the link is a chamber complex.

Proof:
WLOG we can assume A is a face of the fundamental chamber. Then by the correspondence we conclude $A = W_J$.

We have $lk(A) \cong \Sigma_{\geq A}$, meaning the link is isomorphic to the set of standard cosets contained in W_J. That is $\Sigma(W_J, J)$.

$\therefore \; lk(A) \cong \Sigma(W_J, J)$.

-o-

In particular notice that if A & B have the same cotype their link is isomorphic. Using this we give:

Definition: Given a type function τ, we can define the Coxeter Matrix of the apartment as

$$m(s,t) := \operatorname{diam}(lk_{\Sigma} A)$$

with A of cotype $\{s,t\}$.

Proposition: All apartments have the same Coxeter matrix.

Proof: All appartments are isomorphic.

Proposition: If Δ is a building, then so is $lk(A)$ for any $A \in \Delta$. In particular, $lk(A)$ is a chamber complex.

Proof:

Let's pick a system of apartment $\mathcal{A}$ for Δ. Given $A \in \Delta$, consider $lk_\Delta(A)$ & inside of it the subcomplexes $lk_\Sigma(A)$, where $A \in \Sigma \in \mathcal{A}$.

Since Σ is a coxeter simplex, $lk_\Sigma(A)$ is a coxeter simplex itself. Thus if $\mathcal{A}'$ consists of such simplices, it satisfies (B0). Now we justify (B1) & (B2).

(B1): Consider B_1 & B_2 in $lk_\Delta(A)$. Since B_1 & B_2 are joinable to A, there are simplices C_1 & C_2 with $B_1, A \leq C_1$, $B_2, A \leq C_2$.

Apply (B1) to C_1 & C_2 in Δ to find Σ with C_1 & C_2 in it. In particular:

$$A, B_1, B_2 \in \Sigma.$$

Since $C_i \in \Sigma$, then $B_1, B_2 \in lk_\Sigma(A)$, proving

(B1) for A'.

(B2): Consider $lk_{\Sigma_1}(A)$ & $lk_{\Sigma_2}(A)$ be in A' with C_1 & C_2 in them. In particular: $C_1, C_2 \in \Sigma_1$ & Σ_2. Hence there is a simplex B_1 with

$$A, C_1 \leq B_1 \in \Sigma_1$$
$$A, C_2 \leq B_2 \in \Sigma_2$$

⌜We did prove it in class! See the discussion in section 4.1.⌟