Buildings

November 5, 2019

Definition: A W-metric building of type (W, S) is a pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$, whose elements are called chambers, together with a map

$$\delta : \mathcal{C} \times \mathcal{C} \longrightarrow W,$$

called the Weyl distance function, such that for all C, D in $\mathcal{C}$ we have:

(WD1): $\delta(C, D) = 1$ iff $C = D$.

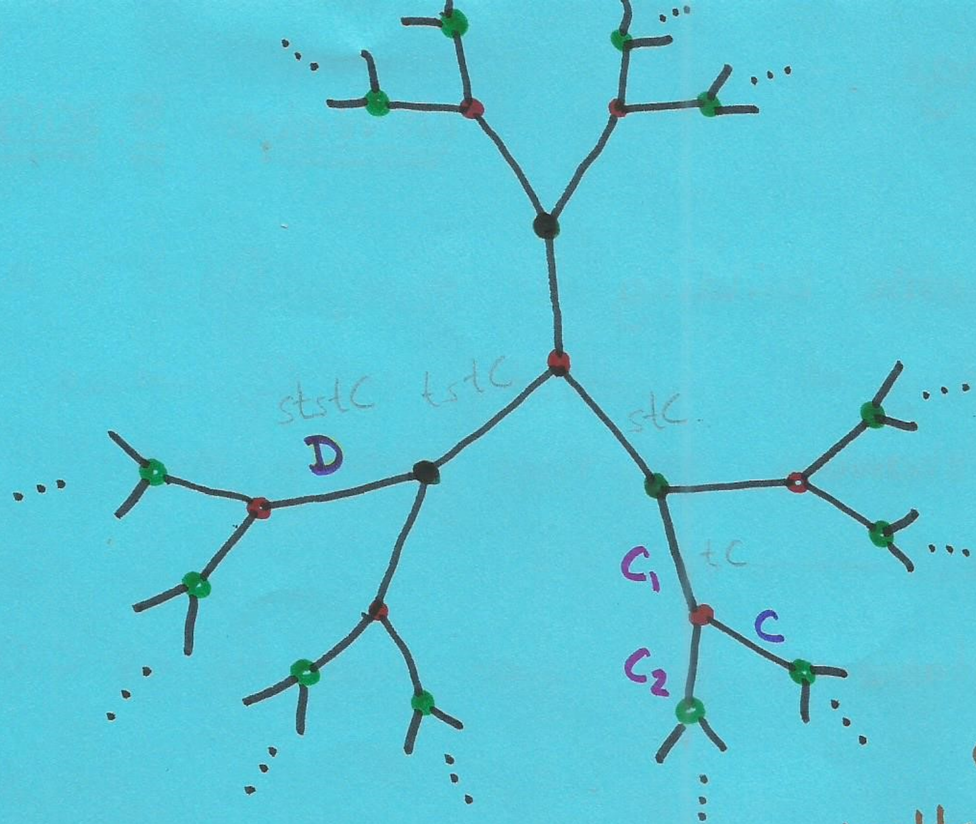
(WD2): If $\delta(C, D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w. If, in addition, $l(sw) = l(w) + 1$, then $\delta(C', D) = sw$.

(WD3): If $\delta(C, D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathcal{C}$ such that $\delta(C', C) = s$ and $\delta(C', D) = sw$.

Let's see some examples. We have seen in previous lecture that a building carries, via its appartements, a W-metric.

Example:

Recall that the building of $SL_2(\mathbb{Q}_2)$ (we have not made this precise) is the following:



It is of type (W,S) where

$$W = \langle s,t \mid s^2 = t^2 = 1 \rangle$$

and the distance is measured with respect to any apartment, which it doesn't matter to compute it.

We are gonna agree that green crossing is on s and red one is a t, then:

$$\delta(C,D) = stst$$

$$\delta(C_1, D) = sts$$

$$\delta(C_2, D) = stst,$$

and notice $\delta(C,C_1) = \delta(C,C_2) = t$, showing both possibilities can occur in (WD 2).

We aim at understanding the properties of buildings from this perspective. ~~We need the following concept:~~