Buildings:

What is our main object to study:

Tits Index:

A Tits index is a triple

$$T = (\Pi, \Theta, A)$$

where:

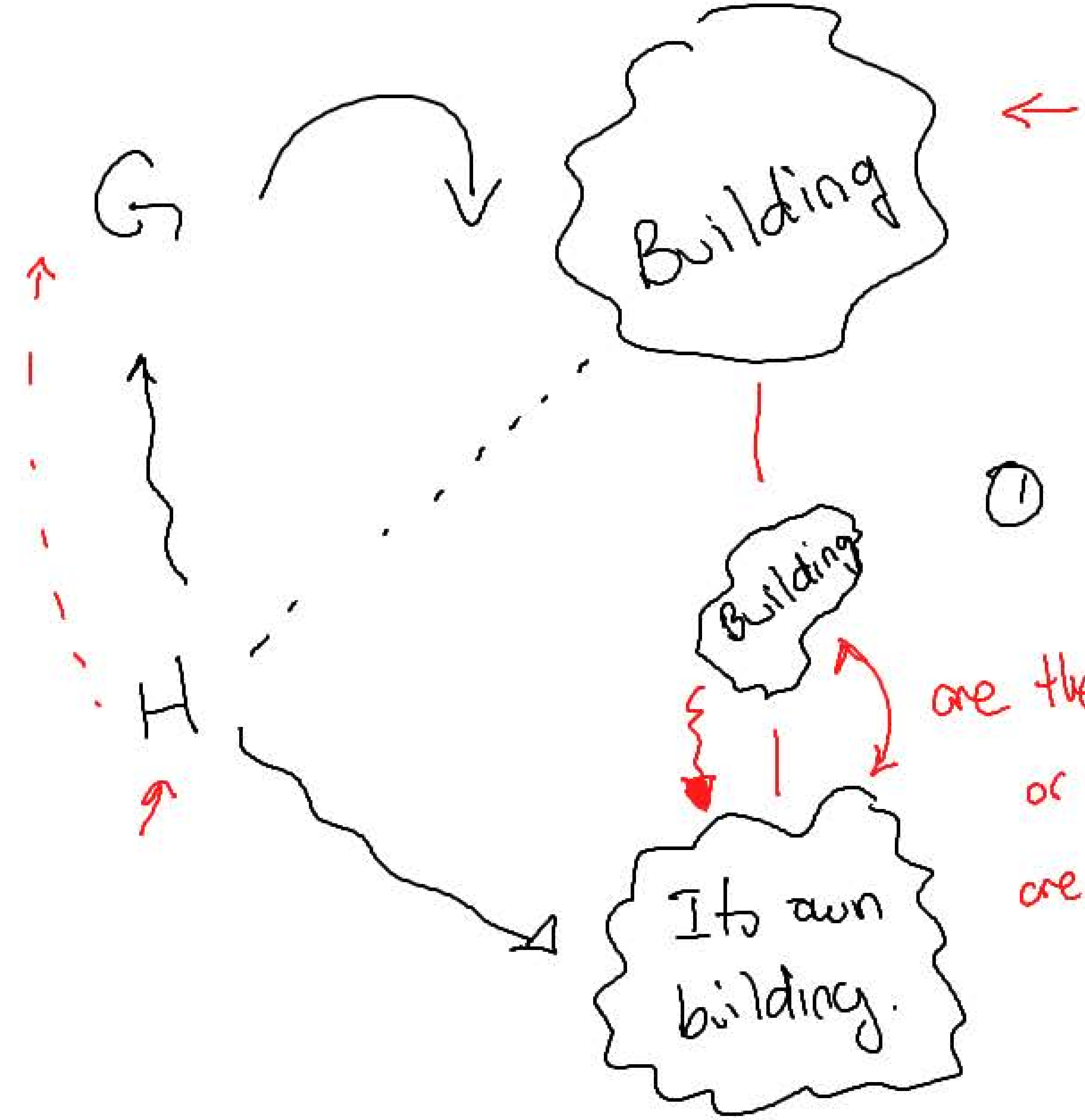
i) A coxeter diagram Π with vertex set S.

ii) Θ is a subgroup of $Aut(W, S)$, where (W, S) is a Coxeter System with generators S.

iii) a subset $A \subseteq S$ is stabilized by Θ s.t. for each $s \in S \setminus A$ the subset $J_s := \Theta(s) \cup A$ is spherical and $w_s A w_s = A$ where w_s is the longest element, & $\Theta(s)$ is the orbit of s.

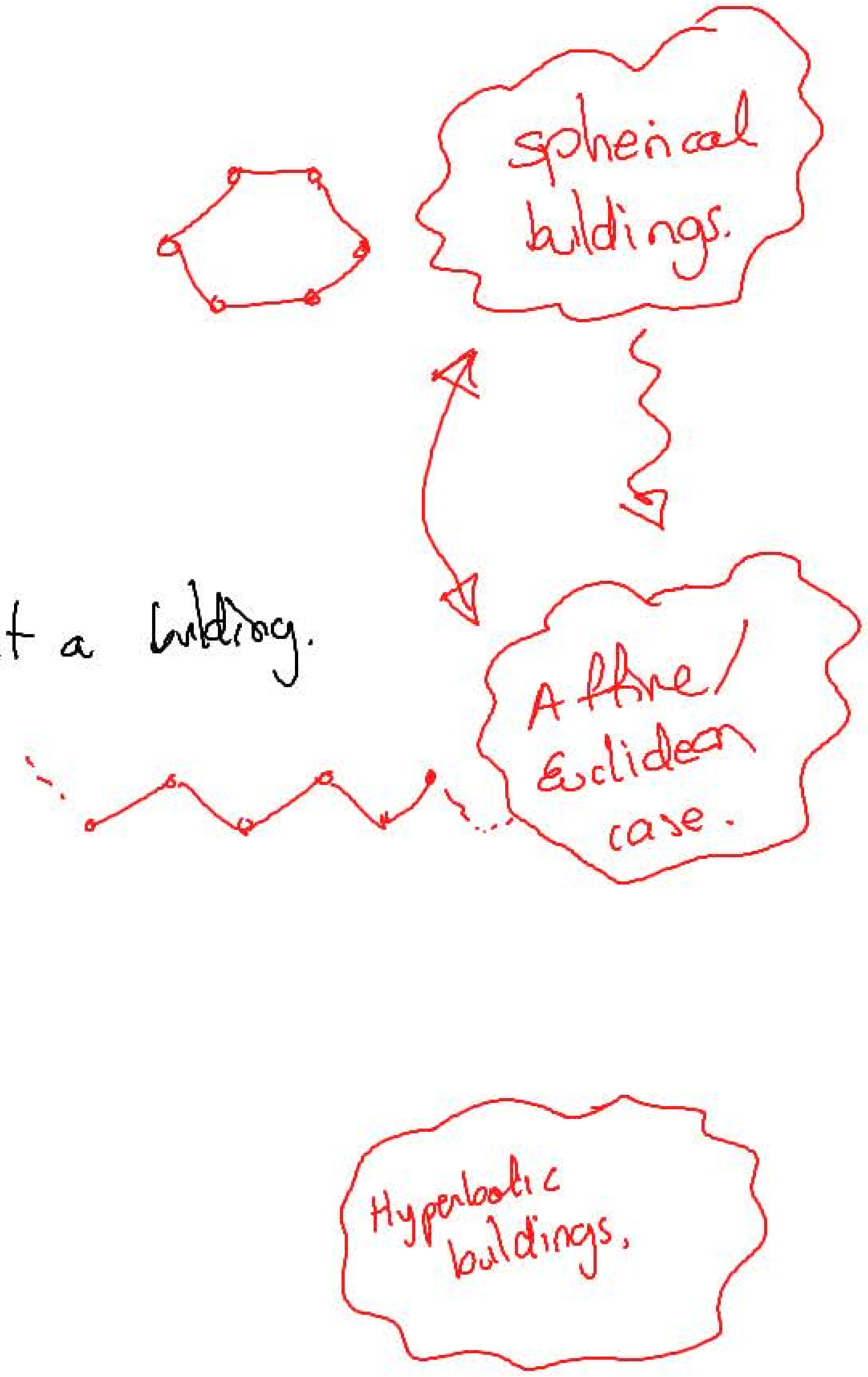
(W, S) is called the absolute type of T.

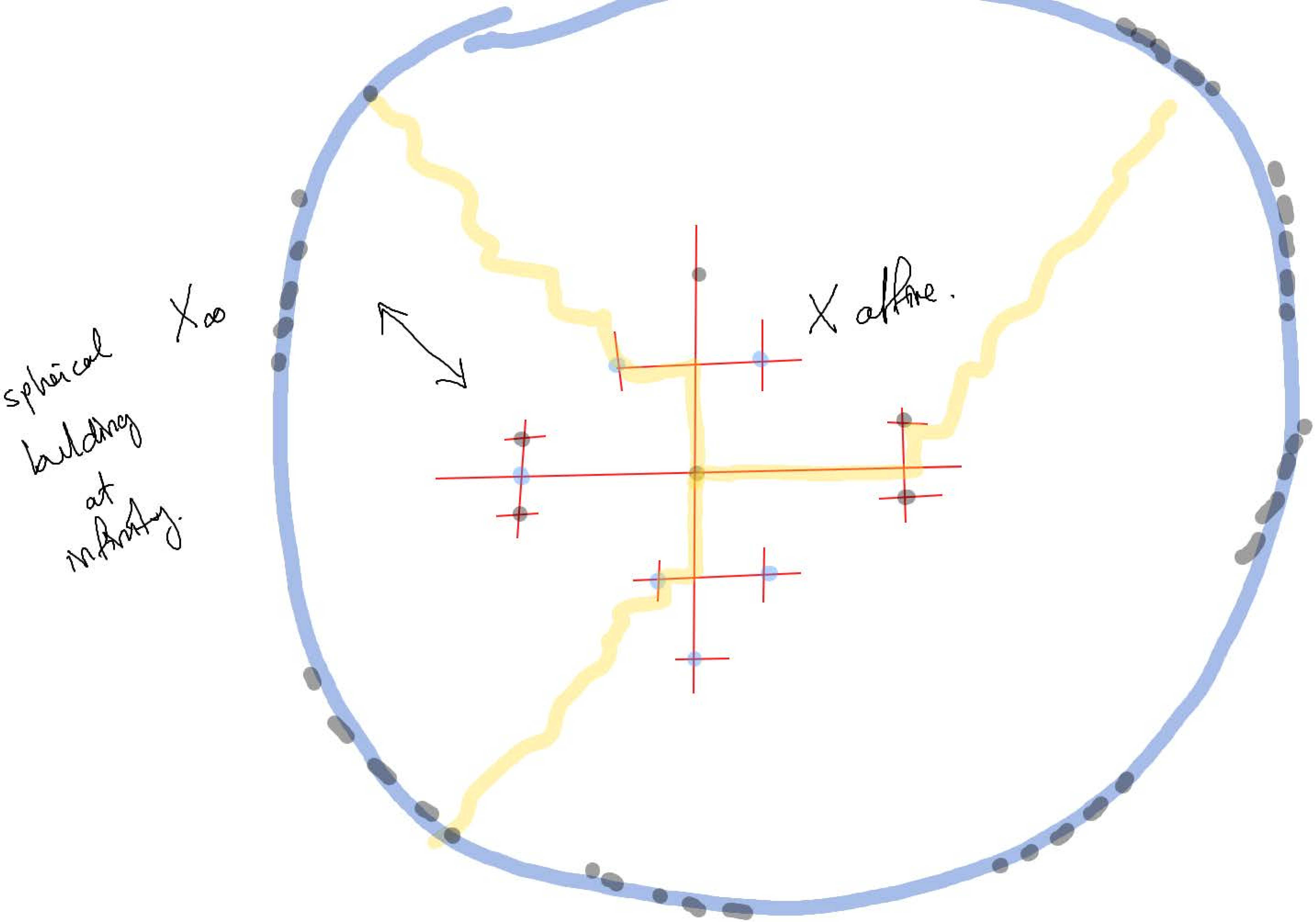
Why do we care about fixed point buildings:



are the same or they are different.

① is the fixed point a building.




spherical building at infinity
X_∞
X affine.

Coxeter Groups:

A Coxeter Group is a pair (W,S) where W is a group generated by the (finite) set S and all elements of S are of order 2, and such that it satisfies any of the following properties:

(C; coxeter condition): For any two elements $s,t \in S$ we have an integeres $m(s,t)$ (possibly infinite) such that

$$W = \langle S \mid (st)^{m(s,t)} = 1 \rangle$$

and there are no further relationships.

(D) If $w = s_1 s_2 \dots s_k$ such that $\ell(w) < k$ then there exists $1 \le i < j \le k$ such that

$$w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_k .$$

(Deletion).

(E): If $w = s_1 \dots s_d$ is a reduced decomp. of w and $s \in S$ then

$$\ell(sw) = \ell(w) + 1$$

or $\exists$ $1 \le i \le d$ such that

$$w = s s_1 \dots \hat{s_i} \dots s_d .$$

(Exchange).

(F): Suppose $w \in W$, $s, t \in S$ are s.t. $\ell(sw) = \ell(w) + 1$, $\ell(wt) = \ell(w) + 1$ then either $\ell(swt) = \ell(w) + 2$ or $swt = w$. (Folding).

(A: action condition): There exist an action

$$\rho : W \times (T \times \{\pm 1\}) \longrightarrow (T \times \{\pm 1\})$$

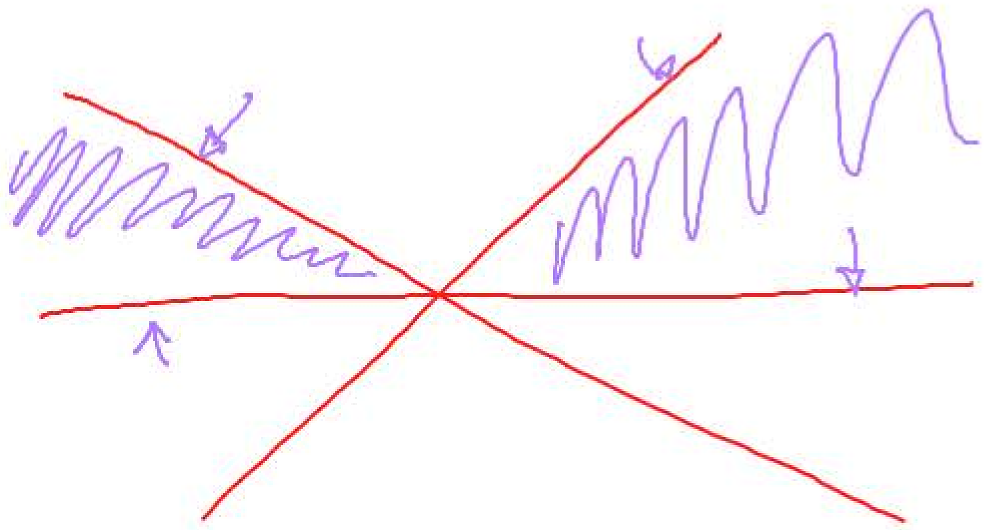
such that it is generated by:

$$\rho_s(w) = \begin{cases} (sws, \epsilon) & s \neq w \\ (s, -\epsilon) & s = w \end{cases} \qquad \forall w \in T,$$

and T consists of all conjugates of elements in S.

Theorem: All conditions are equivalent.

Definition: A coxeter group (W,S) is called spherical if it is finite.



The reason we call this spherical is because we associate a simplicial complex to (W,S) called $\Sigma(W,S)$ and its geometric realization is a sphere.

Originally (i.e. Buildings 1) we constructed a simplex as follows.

1. Defined Parabolic subgroups.
2. " " cosets.
3. we defined a poset of parabolic cosets.
4. Using this POSET we bilt a simplex.

This is a general construction:

$$\text{POSET} \longrightarrow \text{Simplex.}$$

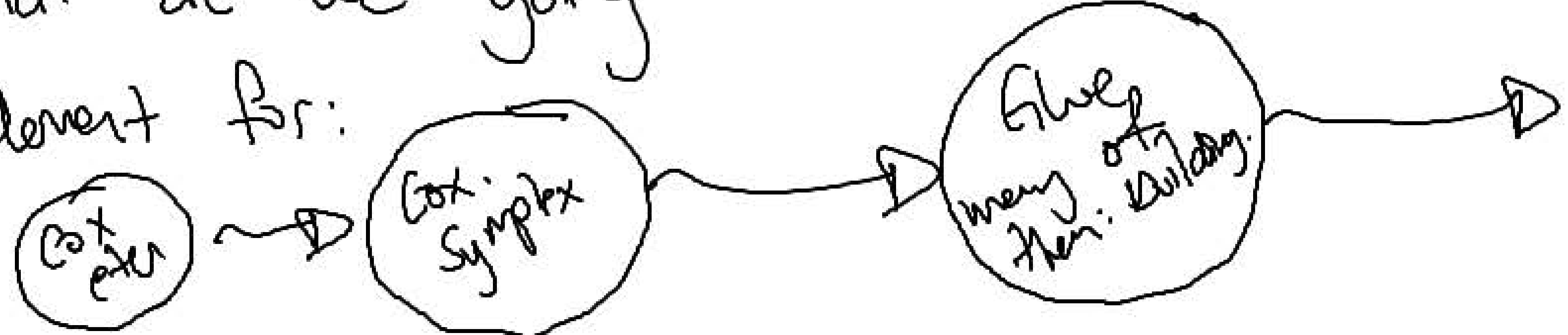
We went to review this in general because later we will repeat this with: "residues".

Certain properties of spherical Coxeter groups:

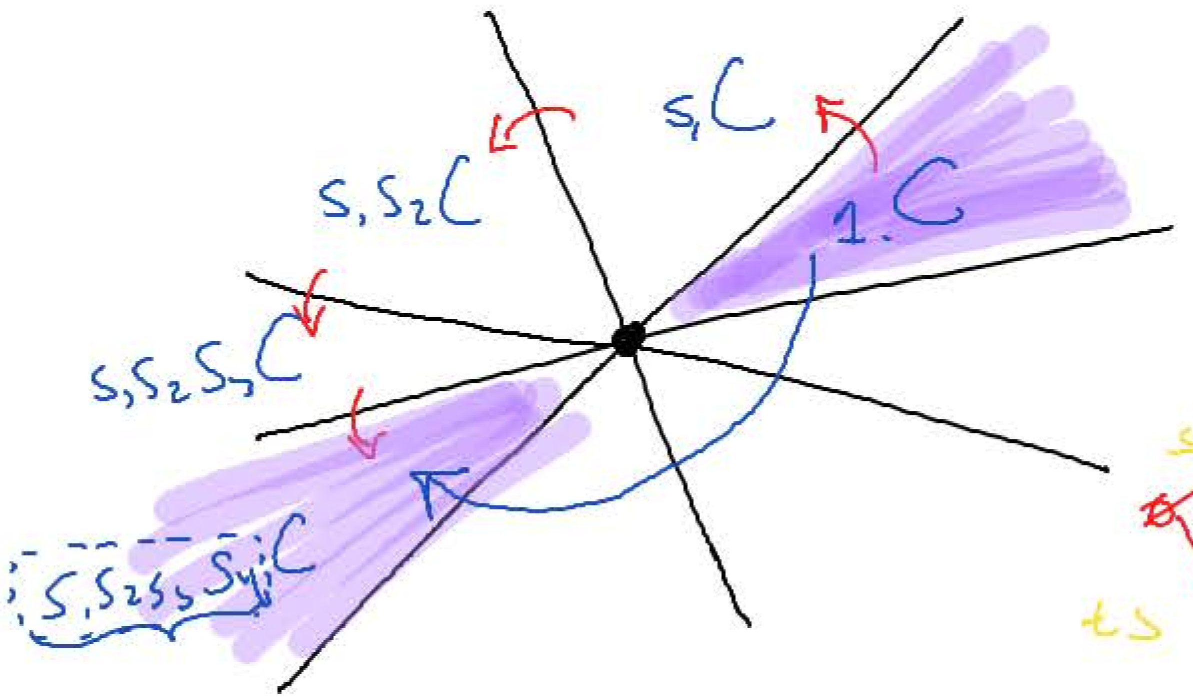
(W,S) spherical coxeter group is associated to a finite hyperplane arrangement.

→ Spherical groups admit a unique longest element w_0.

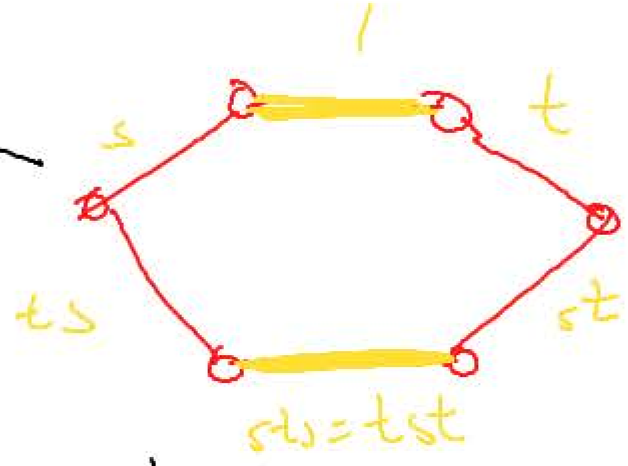
What are we going to use this element for:



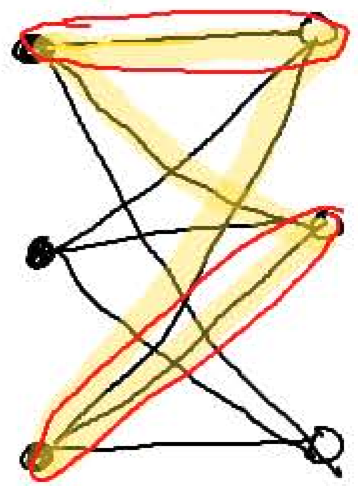
$\mathbb{R}^2$



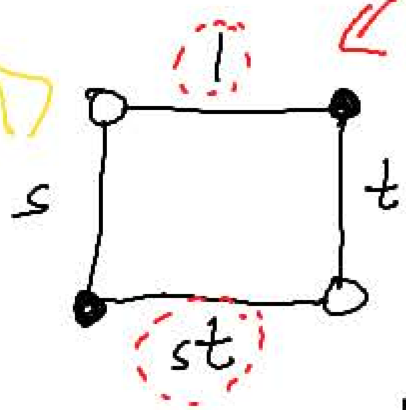
$W=\langle s,t \mid (st)^3=1\rangle$



$K_{3,3}$



$W=\langle s,t \mid (st)^2=1\rangle$



$W=\{1, s, t, st\}$.

$S=\{s,t\}$.

This construction is the opposition involution.

Def: Let (W,S) be a coxeter Group, and $J \subseteq S$ any subgroup.

1) The subgroup $\langle J \rangle \subseteq \langle S \rangle = W$, denoted by W_J, is called a standard parabolic subgroup.

2) (Result): (W_J, J) is a coxeter group.

3) J is spherical if (W_J, J) is spherical as a coxeter group.

(Ex: $\{\{1,s\}, \{s\}\}$ is spherical.) equivalently, W_J is a standard spherical parabolic subgroup.

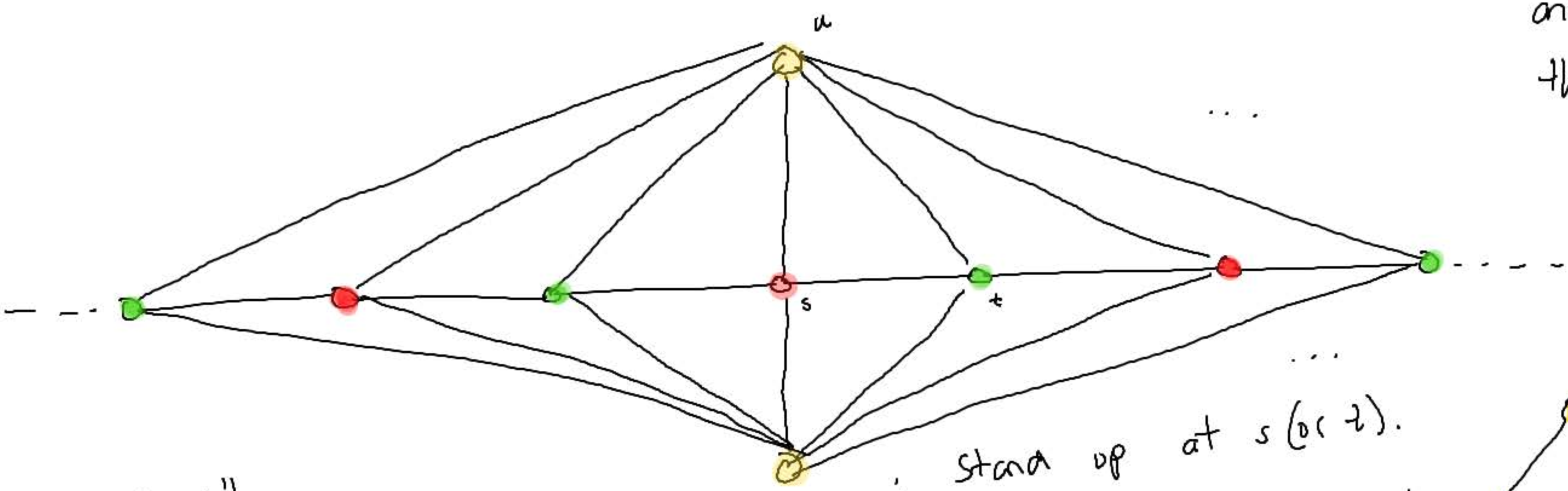
Example:

Let $W = \langle s,t,u \mid s^2=t^2=u^2=1,\ (su)^2=(tu)^2=1 \rangle$.

$S = \{s,t,u\}$.

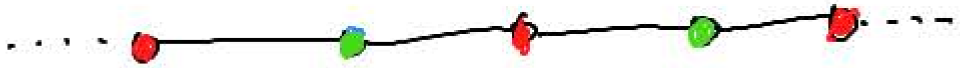
$J = \{s\},\ \{t\},\ \{u\},\ \{s,t\},\ \{s,u\},\ \{t,u\}$

Stand up and look at the link.



Some links are spherical.

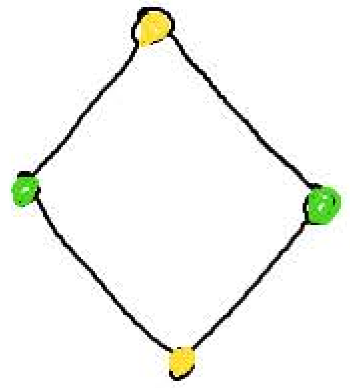
$u \rightsquigarrow$ "Link".



$W_{s,t} = \langle s,t \mid s^2=t^2=1 \rangle$

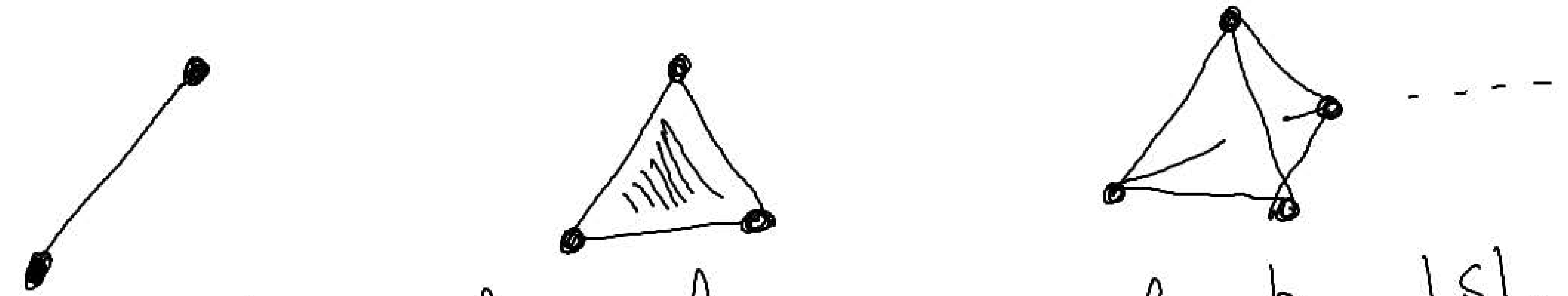
Stand up at s (or t).

$W_{t,u} = \langle t,u \mid t^2=u^2=1,\ (tu)^2=1 \rangle$



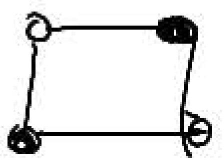
The drawing we have been doing are called Coxeter Complexes.

In order to construct this we begin with a Coxeter group (W,S) and we look at the parabolic cosets (i.e. cosets of standard parabolic subgroups) and we order them by reverse inclusion, and we say they span a face if their intersection is another one of this guys.



The amount of vertices of a face is equal to $|S|$. These guys are glued in very particular way:

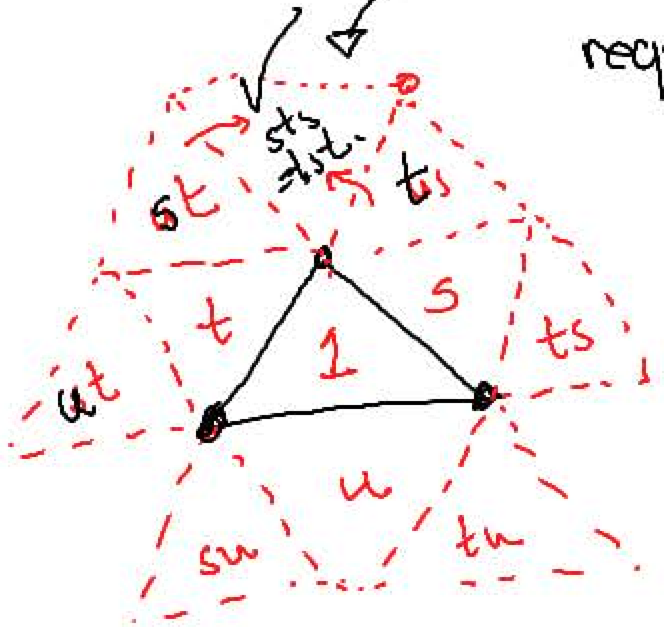
(W,S) has a way to write its element as reduced word:



$$w = s_1 \ldots s_\ell .$$

the idea is that you can move from face to face by following this decomposition:

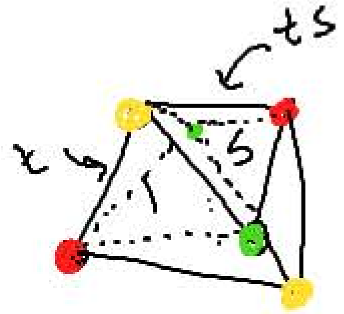
$$W = \langle S \mid (st)^{m(s,t)} = 1 \rangle$$

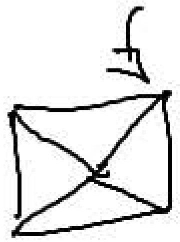


This intersection requires an identity in the group.

$sts = tst.$

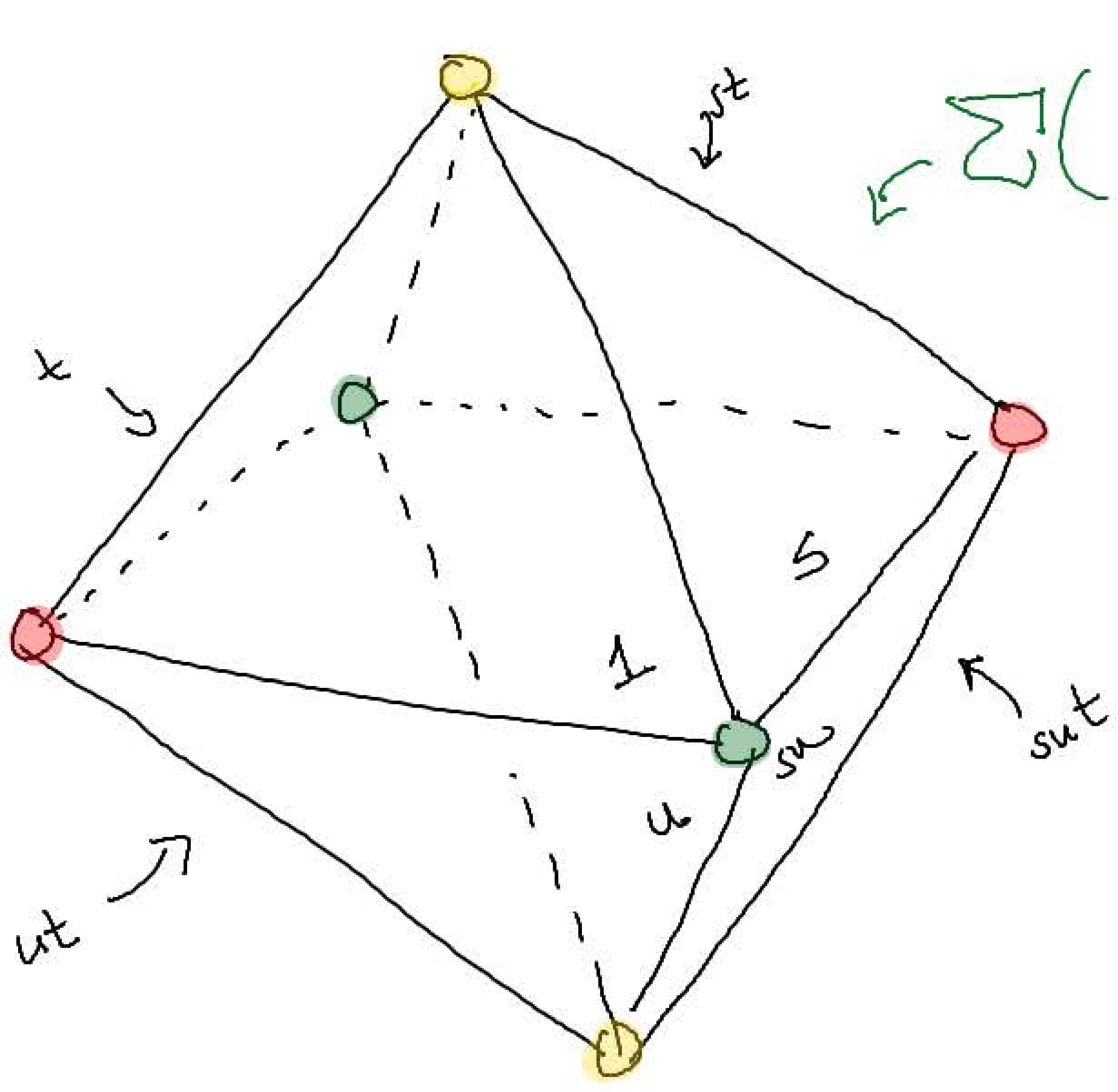
$$W = \left\langle s,t,u \;\middle|\; \begin{matrix} (st)^3 = (su)^3 = (tu)^3 = 1 \\ s^2 = t^2 = u^2 = 1 \end{matrix} \right\rangle$$





$$W = \langle s,t,u \mid s^2 = t^2 = u^2 = (st)^2 = (su)^2 = (tu)^2 = 1 \rangle$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$



$\leftarrow \Sigma\left(s,t,u \mid (st)^2=(su)^2=(tu)^2=s^2=t^2=u^2=1\right)$.

1 & sut correspond to opposite faces.

& sut is the longest element.

<u>Theorem:</u> $\Sigma(W,S)$ actually exists and the faces are in bijection with W. Moreover, this is colorable: one color per element of S.

Descent on building. Mühlherr