The topics we will discuss today are brief since I want to mainly motivate what we will be doing.

Our object of study are buildings and fixed point buildings that appear when we have actions. To study them we need a fundamental tool:

Definition: A Tits Index is a triple

$$\mathcal{I} = (\Pi, \Theta, A)$$

consisting of

i) A Coxeter Diagram Π with vertex set S,

ii) A Subgroup Θ of $Aut(W, S)$, where (W, S) is the Coxeter system corresponding to Π, and

iii) a Subset A of S stabilized by Θ such that for each $s \in S \setminus A$ the subset $J_s := \Theta(s) \cup A$ is spherical and $\omega_s A \omega_s = A$, where $\omega_s := \omega_{J_s}$ is the longest element in W_{J_s} and $\Theta(s)$ denotes the Θ-orbit containing s.

We call (W, S), or sometimes Π, the absolute type of $\mathcal{I}$.

Our first goal will be to review and introduce all concepts required to understand this definition.

These topics (not in order) are:

- What are Coxeter Groups?
- what are Coxeter Diagrams?
- In particular, what is a spherical Coxeter group? What special properties do they have?
- Information about $Aut(W,S)$.
- What is the longest element and which properties does it have and how do we use it?

Mainly, what we need to address first is: spherical coxeter groups and buildings.

We begin with the basic definition:

Definition: Let (W,S) be a pair where W is a group and $S \subseteq W$ is a finite subset of elements of order 2 such that it satisfies:

(c) Coxeter Condition: W admits a presentation

$$\langle S : (st)^{m(s,t)} = 1 \rangle$$

where $m(s,t)$ is the order of st and there is one relation for each pair s,t with $m(s,t) < \infty$.

Such a pair is called a Coxeter System or Coxeter Group.

We have the following result:

Proposition: Let (W, S) be a pair where W is a group and $S \subseteq W$ is a finite subset of elements of order 2. Then the following are equivalent:

- •) Coxeter Condition (C);
- •) Deletion Condition (D): if whenever $w \in W$ has

$$w = s_1 \circ \dots \circ s_d,$$

with $d > \ell_S(w)$ and $s_i \in S$, then there are two indices $1 \leq i < j \leq d$ such that

$$w = s_1 \cdot \dots \cdot \hat{s_i} \cdot \dots \cdot \hat{s_j} \cdot \dots \cdot s_d.$$

- •) Exchange Condition (E): if $w = s_1 \cdot \dots \cdot s_d$ is a reduced decomposition of w, and $s \in S$ then either

$$\ell(sw) = \ell(w) + 1$$

or there exists an index $1 \leq i \leq d$ such that

$$w = s\, s_1 \circ \dots \circ \hat{s_i} \circ \dots \circ s_d.$$

- •) Folding Condition (F): Suppose $w \in W$, $s, t \in S$ are such that $\ell(sw) = \ell(w) + 1$, $\ell(wt) = \ell(w) + 1$. Then either

$$\ell(swt) = \ell(w) + 2$$

or

$$swt = w.$$

- •) Action Condition (A): if there exists an action

$$\rho : W \times (T \times \{\pm 1\}) \longrightarrow (T \times \{\pm 1\})$$

such that in generators $s \in S$ we have

$$\rho_s(w) = \begin{cases} (sws, \epsilon) & s \neq w, \\ (s, -\epsilon) & s = w \end{cases}$$

for all $w \in T$, which is the set of conjugates of S.

Hence, any of the 5 conditions can be used to define a Coxeter System and each one is useful at certain point.

Of special interest for us are the following:

Definition: A coxeter system (W, S) is called spherical if W is finite.

In order to motivate the name we introduce:

Definition: Let (W,S) be a Coxeter System, and let $J \subseteq S$ be any subset. Define

$$W_J := \langle J \rangle \leq W.$$

We call any such W_J a standard parabolic subgroup and the cosets of them are called standard parabolic cosets. (Sometimes we omit the word standard.)

We will review this more carefully later in

the case (we saw it in the first seminar too) 3
is the geometry of the set:

$$\Sigma(W,S) := \{ A \mid A \text{ is a parabolic coset} \},$$

where we furthermore impose:

$$A \leq B \iff B \subseteq A.$$

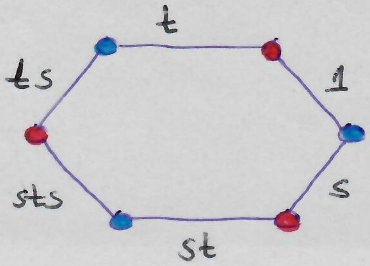
It turns out that like this, $\Sigma(W,S)$ is a POSET out of which a simplicial complex can be built, and whose maximal faces all have the same dimension and are in bijective correspondence with W in a natural way. We call it the Coxeter Complex.

Since we will review this later we dont give precise definitions now but rather focus on examples.

Ex: Consider the group

$$W = \langle s,t \mid s^2 = t^2 = (st)^3 = 1 \rangle$$
$$= \langle 1, s, t, st, ts, sts \rangle$$

The Coxeter Complex is the hexagon:



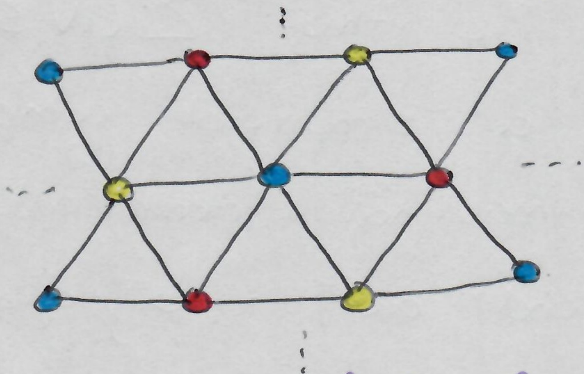
Ex: Consider the coxeter groups:

$$W_1 = \langle s,t,u \mid (st)^3 = (su)^3 = (tu)^3 = s^2 = t^2 = u^2 = 1 \rangle$$

and

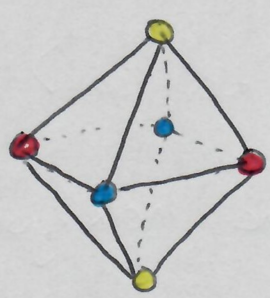
$$W_2 = \langle s,t,u \mid (st)^2 = (su)^2 = (tu)^2 = s^2 = t^2 = u^2 = 1 \rangle.$$

Although both have 3 generators the first one, W_1, is infinite and its coxeter complex is a tesselation:



and we can see the fact of the order of mixed terms being 3 allows us to put hexagons in the plane, since 6 hexagons fit nicely. (We need 6 since we have the equality $sts = tst$, for example.)

For W_2 it is not the same, since now $st = ts$ implies there are 4 triangles, so it cannot be flat, so it curves. In fact it is finite. The complex is:



The reason of the term spherical lies in the following:

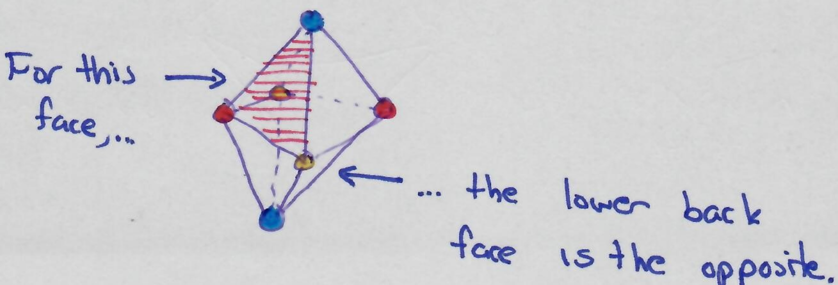
Theorem: The geometric realization of $\Sigma(W,S)$ is compact if and only if it is spherical, in which case $\Sigma(W,S)$ is a triangulation of a sphere.

For example, with W_2 before we get a sphere triangulated as a decahedron.

Spherical Coxeter Groups have a very important property:

Proposition: Let (W,S) be spherical. Then it has a unique element of maximal length which is denoted by w_0.

Geometrically, this element corresponds to paths in the Coxeter Complex from one chamber to the opposite one:



This notion of opposition will be fondamental for us and it already appears on the definition of Tits index. We will study it more carefully as the coulse progresses.