Definition:

Let S be a finite non empty index set and $C = (C_{st})_{s,t \in S}$ be a matrix with entries in $\mathbb{R}$. We say that C is a **Cartan Matrix** if:

(C1) For $s \neq t$ we have $C_{st} \leq 0$ and $C_{st} \neq 0$ iff $C_{ts} \neq 0$.

(C2) We have $C_{ss} = 2$ and, for $s \neq t$, we have

$$C_{st} C_{ts} = 4\cos^2\left(\frac{\pi}{m_{st}}\right),$$

where $m_{st} \geq 2$ is a positive integer or $m_{st} = \infty$.

Consider now a vector space V with $\dim V = |S|$ and a basis given by $\{\alpha_s \mid s \in S\}$. For every $s \in S$ we define a linear transformation:

$$s : V \longrightarrow V$$

defined by, when acting from the right, as:

$$\alpha_t \cdot s = \alpha_t - C_{st}\alpha_s.$$

Notice that in this basis the matrix is:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ C_{ss_1} & C_{ss_2} & \cdots & -1 & \cdots & C_{ss_n} & \\ & & & & \ddots & & \\ & & & & & & 1 \end{pmatrix}$$

The following result is obvious:

Proposition: The linear map $s: V \longrightarrow V$ has the following properties:

$$\alpha_s \cdot s = -\alpha_s, \qquad Tr(s) = |S| - 2, \qquad s^2 = id_V$$

Consequently, s is diagonalizable with $|S|-1$ eigenvalues equal to 1 and one eigenvalue equal to -1.

Proof: —o—

In this way we can regard S as a subset of $GL(V)$ and define:

$$W = W(C) := \langle S \rangle \subset GL(V).$$

Definition: We call $W(C)$ the reflection group associated to C. The subset

$$\Phi = \Phi(C) := \{ \alpha_s \cdot w \mid w \in W, \ s \in S \} \subseteq V$$

is called the root system associated with C

Result: Φ is finite if and only if W is finite.

Proof:

By definition of Φ, W acts on it by permuting the elements. That is, W acts on Φ by permutations.

Suppose then for some $w \in W$,

$$\phi \cdot w = \phi \quad \forall \phi \in \Phi$$

then putting $\phi = \alpha_s$ and varying $s \in S$ we conclude

$$\alpha_s \cdot w = \alpha_s \quad \forall s \in S.$$

$\therefore w = id_V$, since α_s is a basis.

Hence the permutation representation of W on Φ is faithful.

Hence $|W| \leq |S_{|\Phi|}| < \infty$ if Φ is finite.

$\therefore$ Φ finite implies W is finite.

On the other hand, if W is finite then by definition of Φ, as S is finite, it is also finite.

—o—

Example:

Consider the matrix $\begin{pmatrix} 2 & -1/2 \\ -2 & 2 \end{pmatrix}$. We have it is a Cartan matrix. Notice

$$(-2)\cdot\left(-\tfrac{1}{2}\right) = 1 = 4\cos^2\left(\frac{\pi}{m_{st}}\right),$$

where $\frac{\pi}{m_{st}} = \frac{\pi}{m_{ts}} = \frac{\pi}{3}$, i.e. $m_{st} = 3$.

Consider $\mathbb{R}^2$ with the standard basis $e_1 = \alpha_s = (1,0)$ & $e_2 = \alpha_t = (0,1)$. We have then that

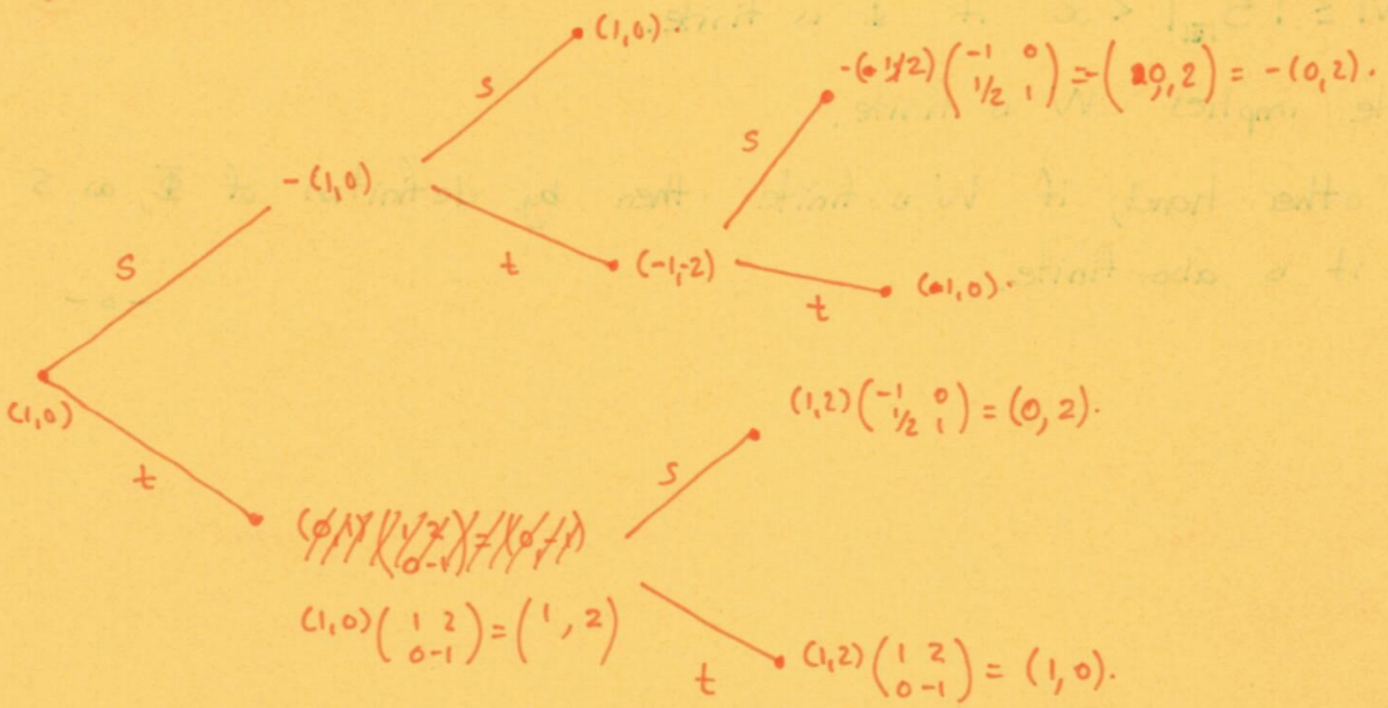
$$[s] = \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \qquad [t] = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

That is, for example,

$$\alpha_s \cdot s = (1,0)\begin{pmatrix} -1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} = (-1,0) = -\alpha_s.$$

$$\alpha_t \cdot s = (0,1)\begin{pmatrix} -1 & 0 \\ 1/2 & 1 \end{pmatrix} = (\tfrac{1}{2},1) = \tfrac{1}{2}\alpha_s + \alpha_t = \alpha_t - C_{st}\alpha_s$$

Analogously for α_t. Let us obtain Φ in a "brute force" way:



From here we get: $\pm(1,0)$, $\pm(1,2)$, $\pm(0,2)$.

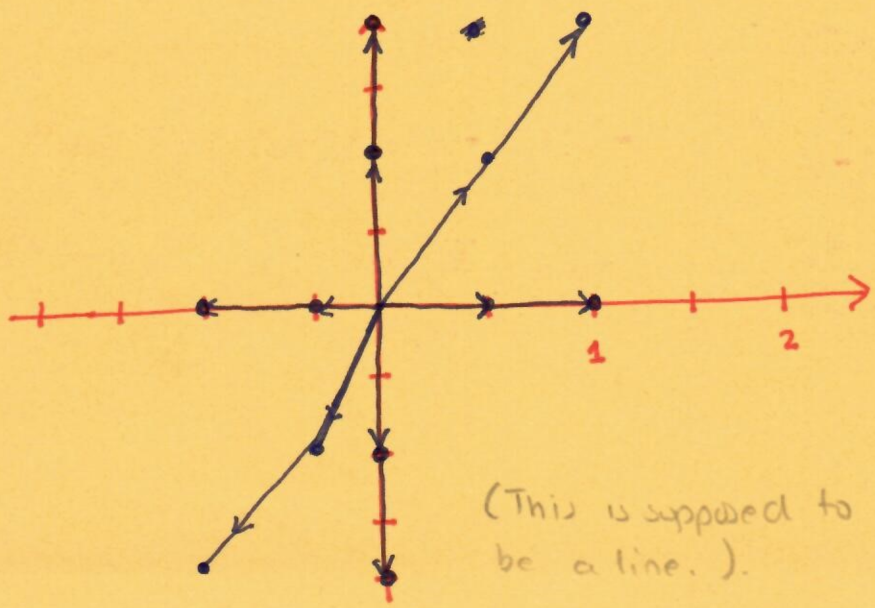
Repeating the same we get: $\pm(0,1)$, $\pm(1/2,1)$, $\pm(1/2,0)$.

Checking these trees we find $sts = tst$ and all other ones are different, hence:

$$W = \langle 1, s, t, st, ts \quad sts = tst \rangle.$$

$$\Phi = \{ \pm(1,0), \pm(1,2), \pm(0,2), \pm(0,1), \pm(1/2,1), \pm(1/2,0) \}$$

which can be drawn as:



Proposition: Let $s,t \in S$, $s \neq t$ and $V_{st} := \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t \subseteq V$. Then V_{st} is invariant under s & t. The action of s & t on V_{st} with respect to the basis $\{\alpha_s, \alpha_t\}$ is:

$$M_s = \begin{pmatrix} -1 & 0 \\ -C_{st} & 1 \end{pmatrix}, \quad M_t = \begin{pmatrix} 1 & -C_{ts} \\ 0 & -1 \end{pmatrix}.$$

Furthermore, if $m_{st} < \infty$, there exists a unique subspace $\mathcal{U} \subseteq V$ such that $V = V_{st} \oplus \mathcal{U}$ and each of s,t acts as the identity on $\mathcal{U}$.

Proof: —o—

Given $s,t \in S$, $s \neq t$ we can define $W_{s,t} := \langle s,t \rangle \subseteq W$ and $\Psi := \{\alpha_s \cdot w', \alpha_t \cdot w' \mid w' \in W_{s,t}\}$. Since s,t are of order 2 the group $W_{s,t}$ is a dihedral group. The structure result is:

Theorem: We have that:

a) the element $st \in W_{st}$ has order m_{st}.

b) for each $\alpha \in \Psi$, we have $\alpha \geq 0$ or $\alpha \leq 0$, with respect to α_s, α_t.

c) If $\alpha \in \Psi$ and $u \in \{s,t\}$ are such that $\alpha \geq 0$ but $\alpha \cdot u \leq 0$ then α is a multiple of α_u.

These two rank subgroups are important as we reduced many problems to them by induction many times. The main result of these groups is:

Theorem: W admits the presentation:

$$W = \langle S \mid s^2 = 1 \ \forall s \in S \text{ and } (st)^{m_{st}} = 1 \ \forall s \neq t,\ m_{st} < \infty \rangle.$$

Hence we have that a Coxeter matrix constructs a Coxeter Group.

Ofcours a Coxeter Group constructs a Coxeter matrix by defining

$$\cancel{C_{st}/4=-4\cos^2(\pi/m_{st})}\quad C_{st} = -2\cos\left(\frac{\pi}{m_{st}}\right).$$

where m_{st} is the order. The reflection group then coincides with W since they have the same representation.

Definition: We say a Cartan Matrix is decomposable if $S = S_1 \sqcup S_2$ where $C_{st} = 0$ for all $s \in S_1$, $t \in S_2$. If not such S_1, S_2 exist we say it is idecomposable.

~~I~~decomposability breaks up the carton matrix, maybe after reordering, into blocks

$$C = \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right)$$

and $V = V_1 \oplus V_2$ where $V_1 = \mathbb{R}^{S_1}$, $V_2 = \mathbb{R}^{S_2}$ and moreover

$$\alpha_s \cdot t = \alpha_s - C_{ts}\alpha_t = \alpha_s \qquad \text{if } s \in S_i,\ t \in S_j.$$

Hence $\overset{s}{\cancel{\alpha_s}}$ & $\overset{t}{\cancel{\alpha_t}}$ commute with each other and we can break up:

$$W \cong W_1 \times W_2$$
$$w \mapsto (w|_{V_1}, w|_{V_2})$$

and $\Phi = \Phi_1 \sqcup \Phi_2$ where $\Phi_1 = \Phi \cap V_1$, $\Phi_2 = \Phi \cap V_2$. And $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

So we can reduce the study of Cartan matrices to idecomposable ones.

Def: A Coxeter Group (W,S) with Cartan matrix that is idecomposable is called irreducible.

Proposition: (W,S) is irreducible if and only if there do not exists $S_1, S_2 \neq \emptyset$, $S = S_1 \sqcup S_2$ with $(s_i s_j)^2 = 1$ $\forall s_i \in S_1, s_j \in S_2$.

Proof: This follows from the fact that $C_{st} = 0$ iff $m_{st} = 2$.

-o-

Let us now suppose that for a cartan matrix C and its reflection group $W(C)$ there exists a W-invariant symmetric bilinear form $\langle \, , \rangle$ with $\langle \alpha_s, \alpha_s \rangle > 0$ $\forall s \in S$.

Then

$$\langle \alpha_t \cdot s, \alpha_s \cdot s \rangle = \langle \alpha_t - c_{st}\alpha_s, -\alpha_s \rangle$$
$$= \langle \alpha_t, -\alpha_s \rangle + c_{st} \langle \alpha_s, \alpha_s \rangle$$

but $\langle \alpha_t \cdot s, \alpha_s \cdot s \rangle = \langle \alpha_t, \alpha_s \rangle$, then

$$\boxed{\frac{2\langle \alpha_t, \alpha_s \rangle}{\langle \alpha_s, \alpha_s \rangle} = c_{st}.}$$

and we can call $d_s^2 = \langle \alpha_s, \alpha_s \rangle$, $d_s > 0$, we obtain:

$$d_s^2 c_{st} = 2 \langle \alpha_s, \alpha_t \rangle = d_t^2 c_{ts}.$$

$$\therefore \quad \frac{d_s d_t^{-1} c_{st}}{2} = -\cos\left(\frac{\pi}{m_{st}}\right).$$

Conversely, if we find positive ~~real~~ numbers with

$$\frac{d_s d_t^{-1} C_{st}}{2} = -\cos\left(\frac{\pi}{m_{st}}\right),$$

then defining

$$(\alpha_s, \alpha_t) = \frac{d_s^2 C_{st}}{2},$$

we obtain an invariant product:

$$\begin{aligned}(\alpha_s \cdot u, \alpha_t \cdot u) &= (\alpha_s - C_{us}\alpha_u, \alpha_t - C_{ut}\alpha_u) \\ &= (\alpha_s, \alpha_t) - \frac{C_{us} d_u^2 C_{ut}}{2} - \frac{C_{ut} d_s^2 C_{su}}{2} + d_u^2 C_{us} C_{ut} \\ &= (\alpha_s, \alpha_t)\end{aligned}$$

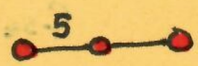
Ex: If C is a symmetric matrix then put $d_t = 1$ $\forall t \in S$.

Definition: A cartan matrix is of finite type if there exists such a bilinear form that is also positive definite.

In order to classify them we build a graph: given a Cartan matrix $C = (C_{st})_{s,t\in S}$ we obtain the (m_{st}) and build a graph with one vertex for each $s \in S$ and a edge joining s & t if $m_{st} > 2$. In case $m_{st} > 3$ we label the edge with m_{st}.

Theorem: The Coxeter Graphs of idecomposable Cartan matrices of finite type are given by:

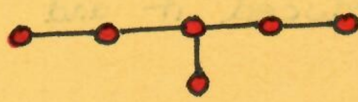
A_n: 
$n \geq 1$

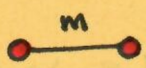
H_3: 

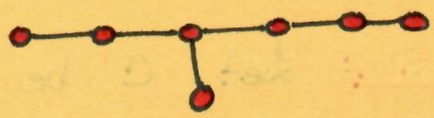
B_n: 
$n \geq 2$

H_4: 

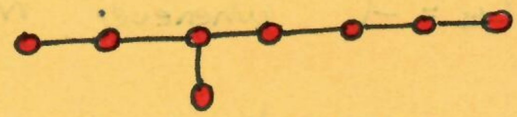
D_n: 
$n \geq 4$

E_6: 

$I_2(m)$: 
$m \geq 5$

E_7: 

F_4: 

E_8: 

(The subindex corresponds to the rank.)

It may happen that different Cartan Matrices Give the same Coxeter Graph, and so that different ~~Cos~~ Cartan matrix give ~~a~~ the same Coxeter Group. We would like to single out a "better" Cartan matrix.

Definition: We say that Φ is a reduced root system if, for any $\alpha \in \Phi$, and $x \in \mathbb{R}$ we have $x\alpha \in \Phi$ implies $x = \pm 1$.

Every coxeter group (W, S) arises from a cartan matrix whose root system is reduced, for example define

$$c_{st} = -2\cos\left(\frac{\pi}{m_{st}}\right).$$

and put $d_s = 1$.

We have the following result:

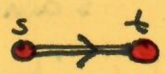
Proposition: Assume C is of finite type. Then the root system Φ is reduced if and only if $c_{st} = c_{ts}$ for all s, t with m_{st} odd.

This motivates the following:

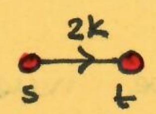
Definition: Let C be a cartan matrix. We say C is **standard** if $c_{st} = c_{ts}$ whenever m_{st} is odd and $c_{ts} = -1$ or $c_{ts} = -1$ whenever $m_{st} \geq 4$ is even.

Once we have a standard cartan matrix we modify the Coxeter Graph by the following rule:

- we add an arrow to the label between s & t, if there is one according to:

- m_{st} odd   no label
- $m_{st} = 4$ 

   $c_{st} = -1$
- $m_{st} = 6$ 

   $c_{st} = -1$
- $m_{st} = 2k \geq 8$ 

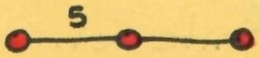
   $c_{st} = -1$

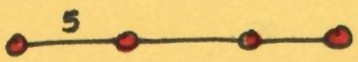
The arrow points to the element of smaller norm.

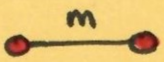
The graph obtained is called the Dynkin Diagram.

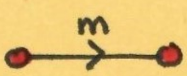
Definition: Let C be a indecomposable Cartan Matrix. C is **crystallographic** if it is of finite type and all its entries are rational integers. In such a case we call $W = W(C)$ a **Weyl Group**. Otherwise, ~~the~~ we say C is of **noncrystallographic type**. 6

The idecomposable non crystallographic Dynkin diagrams are:

H_3: 

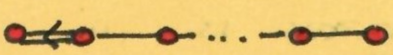
H_4: 

$I_2(m)$, $m \geq 5$ odd 

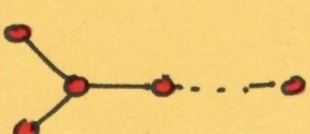
$I_2(m)$, $m \geq 8$ even 

Those of crystallographic type are:

A_n: $n \geq 1$ 

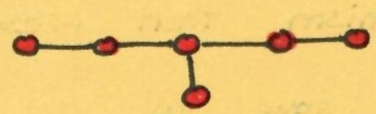
B_n: $n \geq 2$ 

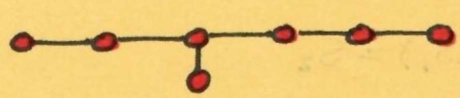
C_n: $n \geq 3$ 

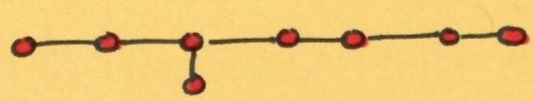
D_n $n \geq 4$ 

$G_2 = I_2(6)$ 

F_4: 

E_6: 

E_7: 

E_8: 

Proposition: Let C be a Cartan matrix and (W,S) be the corresponding Coxeter System. Then W is finite if and only if C is of finite type.

This gives motivation to understand the structure of finite coxeter groups. The information to create the coxeter graph can be encoded in a matrix too:

Definition: Let S be an indexing finite set. A Coxeter Matrix is a matrix $M = (m_{st})_{s,t \in S}$ such that:

i) m_{st} is a positive integer or ∞.

ii) $m_{st} = m_{ts}$ $\forall s \neq t$ in S, and $m_{st} \geq 2$ $s \neq t$.

iii) $m_{ss} = 1$

Of course, we can build a coxeter matrix out of a cartan matrix via the relation

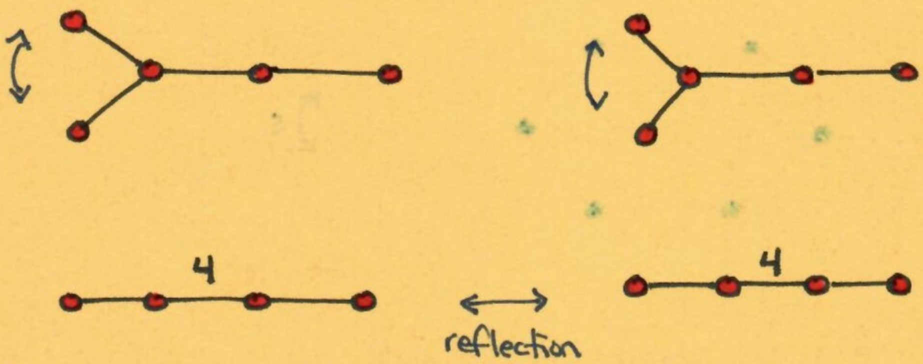
$$c_{st} = -2\cos\left(\frac{\pi}{m_{st}}\right),$$

and all the work before shows that a ~~cartan~~ coxeter matrix leads to a unique diagram and viceversa (modulo permutation)

A concept that now becomes central is the following:

Definition: An isomorphism of Coxeter Diagrams is a graph isomorphism that preserves the edge labels. Of course this is the same as a group isomorphism $\phi: (W_1, S_1) \longrightarrow (W_2, S_2)$ with $\phi(S_1) = S_2$

The reason being that if we have such a graph map



we can consider the map from the Free Group

$$\tilde{\phi}: \langle S_1 | - \rangle \longrightarrow (W_2, S_2)$$
$$s_i \longmapsto t_i$$

where t_i is the labeling, and because we preserve the labels

$$\tilde{\phi}((s_i s_j)^{m_{ij}}) = (\tilde{\phi}(s_i)\tilde{\phi}(s_j))^{m_{ij}}$$
$$= (t_i t_j)^{m_{ij}}$$
$$= 1.$$

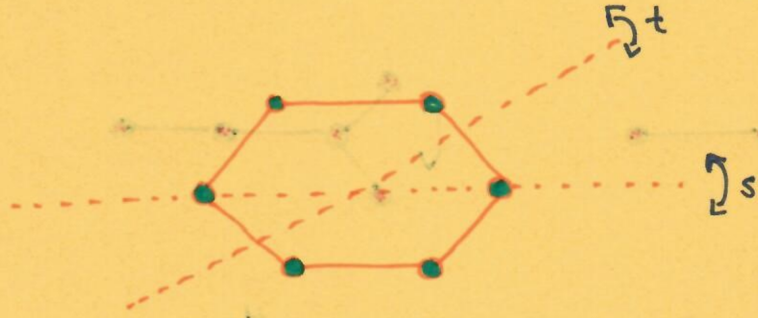
$\therefore$ By the Coxeter Condition of (W_1, S_1) we have

$$\phi : (W_1, S_1) \longrightarrow (W_2, S_2)$$
$$\phi(S_1) = S_2.$$

An important kind of elements, which already appeared in condition (A), are called the reflections:

<u>Definition:</u> Let (W, S) be a Coxeter Group. The conjugates of elements in S are called reflections.

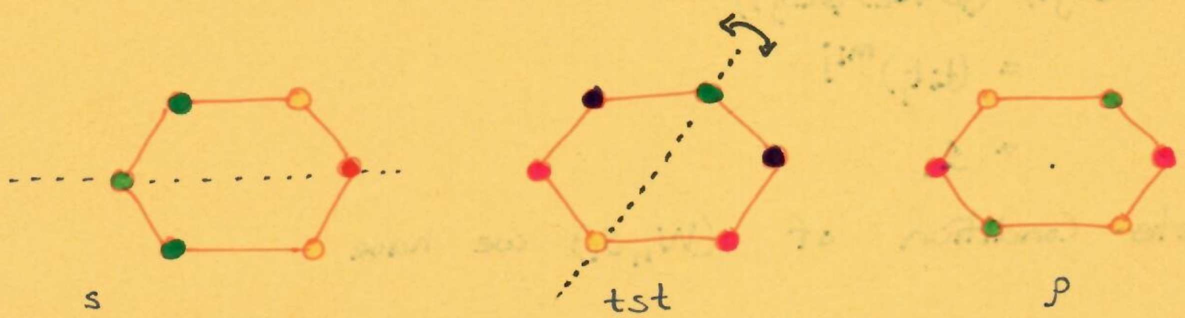
Example: Consider a hexagon and some symmetries:



and let ρ be the central symmetry. Let us call W the set of ~~rotations~~ symmetries of our hexagon, i.e. a Dyhedral group.

We know $S_1 = \{s, t\}$ is a set of generators and that (W, S_1) is a Coxeter Group.

Now consider $S_2 = \{s, tst, \rho\}$. We can see what happens to the vertices with each one:



It is clear that $\rho = tststs$ and so

$$s(tst)\rho s = stst\,tststs\,s$$
$$= t,$$

so these generate W as well. We can prove that actually (W, S_2) is also a coxeter ~~diag~~ system! With the first coxeter system we find a presentation

$$W = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle$$

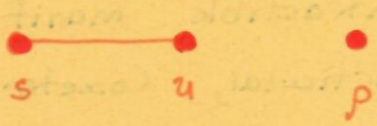
and with the other we get; letting: $u = tst$,

$$W = \langle s, u, \rho \mid s^2 = u^2 = \rho^2 = (su)^3 = (s\rho)^2 = (u\rho)^2 = 1 \rangle.$$

Hence the first one is of Coxeter Graph:

$I_2(6)$.

while the second one is of coxeter diagram



$A_2 \times A_1$.

Hence W, ~~despite having the same Coxeter Graph~~ can be part of a Coxeter System in two different non isomorphic ways.

-o-

This leads to the following:

Definition: Let G be a group and $R \subseteq G$ a set of involutions. We say R is a Coxeter generating set for G if (G, R) is a coxeter system (in particular, $\langle R \rangle = G$).

With this at hand we can define:

Definition: Let M be a coxeter matrix (or diagram). We say M is rigid if all coxeter generating sets of $W(M)$ have the same coxeter graph. We say it is strongly rigid if all coxeter generating sets are conjugate in $W(M)$.

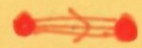
In this direction we have:

Theorem: We have:

1.- (D. Radcliffe): If all labesl of $\Gamma(M)$ are ∞ then M is rigid.

2.- (R. Charney & M. Davis): Let (W,S) be ~~strongly~~ a Coxeter System. If W is capable of acting effectively, properly and cocompactly on some contractible manifold, then (W,S) is strongly rigid. In particular, Coxeter Groups of affine & compact hyperbolic type are strongly rigid.

3.- Suppose (W,S) is irreducible, non-spherical and 2-spherical (i.e. no label is ∞), then (W,S) is strongly rigid.

Notice our previous example doesn't contradict this since $I_2(6)$ is spherical as it is christallographic with Dynkin diagram 

The importance of these concepts is that in the case of rigidity we can answer whether, ~~that~~ given two Coxeter matrices M_1 & M_2 the groups $W(M_1)$ & $W(M_2)$ are isomorphic by looking at the graphs.

In the case of strong rigidity it is even better since all isomorphisms come from a conjugation followed by a graph isomorphism!