We say a Coxeter Group is of classical type if it is of type A_n, B_n, C_n, D_n. We give a brief tour through them in this section:

Definition: Let $n \geq 1$ and $W_n \subseteq GL_n(\mathbb{R})$ be the subgroup consisting of all matrices which have precisely one non-zero entry in each row and each column, and where this non zero entry is ± 1.

Fact: W_n is a finite group with $2^n \cdot n!$ elements.

Clearly we can also define the equivalent subgroup that only has $+1$, instead of ± 1. Call this H_n. Finally call N_n the diagonal matrices in W_n.

Fact: $H_n \cdot N_n = W_n$ as a "direct" product.

Proof: We have

$$|H_n||N_n| = n! \cdot 2^n = |W_n|,$$

and since

$$|H_n \cdot N_n| = \frac{|H_n||N_n|}{|H_n \cap N_n|} = |H_n||N_n|,$$

since $H_n \cap N_n = \{1\}$ clearly, we deduce $H_n \cdot N_n = W_n$. -o-

More precisely, H_n acts by conjugation on N_n and so we have

$$W_n = N_n \rtimes H_n$$

H_n can be identified with S_n, and in S_n we have a well known set of generators: $s_1, \dots, s_{n-1}$ where s_i

permutes $(i, i+1)$. Identifying s_i with the matrix in H_n we have s_i generate H_n.

Finally, let t_i, $i=0,\dots,n-1$ be the diagonal matrix with only a -1 in the $(i+1)$th entry and 1 in the others.

Proposition: $s_1,\dots,s_{n-1}$ and ~~$t_0,\dots,t_n$~~ $t=t_0$ generate W_n. They satisfy

- $ts_1ts_1 = s_1ts_1t$,
- $ts_i = s_it, \quad i>1$
- $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$,
- $s_is_j = s_js_i, \quad |i-j|>1$

Succintly: $(ts_1)^4=1$, $\underset{i>1}{(ts_i)^2=1}$, $(s_is_{i+1})^3=1$, $\underset{|i-j|>1}{(s_is_j)^2=1}$. Moreover any $t_1,\dots,t_{n-1}$ is conjugate to t.

Proof: The relationships can be verified easily. We only write

$$t_i = s_is_{i-1}\dots s_1ts_1\dots s_{i-1}s_i, \quad 1\le i\le n-1,$$

to see the conjugation.

—o—

If we put the order $t, s_1, s_2, \dots, s_{n-1}$ we obtain that W_n is a quotient of the coxeter system of type B_n:

what we haven't seen is that these are defining relationships. We will do this just to show an example of how this works: To begin with, consider $\mathbb{R}^n$ with its standard basis $e_1,\dots,e_n$ and standard inner product.

Proposition: Let $n \geq 2$ and $W_n = \langle S \rangle \leq GL_n(\mathbb{R})$. Define the matrix $C = (c_{st}) = (c_{ij})_{0 \leq i,j \leq n-1}$ by:

$$c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

where $\alpha_0 = e_1$, $\alpha_1 = e_2 - e_1$, $\dots$, $\alpha_{n-1} = e_n - e_{n-1}$. Then C is the Cartan matrix given by the Dynkin diagram

of type B_n, with generators $t, s_1, \dots, s_{n-1}$, respectively. Thus, up to a change of basis of $\mathbb{R}^n$, the group $W_n \subseteq GL_n(\mathbb{R})$ is the reflection group associated to a Cartan matrix of type B_n.

Proof: What need to be checked is that $R_t, R_{s_1}, \dots, R_{s_{n-1}}$ are exactly $t, s_1, \dots, s_{n-1}$, where $R_t, \dots, R_{s_{n-1}}$ are the reflections. Basically we need the relationships in the basis. Pick, for example,

$$t = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and let's see its action on $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$:

$$\alpha_0 \cdot t = (1, 0, \dots, 0)\begin{pmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = -\alpha_0.$$

$$\alpha_1 \cdot t = (e_2 - e_1) \cdot t = e_2 \cdot t - e_1 t = e_2 + e_1 = (e_2 - e_1) - (-2e_1)$$

$$\alpha_k \cdot t = (e_k - e_{k-1}) \cdot t = e_k - e_{k-1} = (e_k - e_{k-1}) - 0 \cdot e_1,$$

which are the defining relationships of R_t. In analogous way we deal with $s_1, \dots, s_{n-1}$, only noticing that:

$$c_{ii} = 2, \qquad c_{i,j} = c_{j,i} = 0 \text{ if } |i - j| \geq 2,$$

$$c_{i,i+1} = c_{i+1,i} = -1, \; i \geq 1, \quad c_{1,0} = -1, \; c_{0,1} = -2.$$

$\therefore W_n = W(C)$ for this C, which is a cartan matrix, and for the system $\alpha_0, \dots, \alpha_{n-1}$.

$\therefore W_n$ is a Coxeter group with given ~~coxeter~~ set of generators.

—o—

Notice that we already know W_n is finite, hence the roots have to be finite too. We know that

$$C_{ij} = C_{ji} \quad \text{for } m_{ji} \text{ odd,}$$

as this only happen for $m_{i,i+1} = 3$ $i \geq 1$. Hence the ~~Coxeter~~ root system is reduced.

Proposition: For each $n \geq 2$ we have

$$\Phi_n^+ = \{ e_j \pm e_i \mid 1 \leq i < j \leq n \} \cup \{ e_i \mid 1 \leq i \leq n \}$$

Thus there are n^2 positive roots in a reduced system of type B_n.

Proof: We have to compute the orbit of the elements $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$

- For α_0: Since $\alpha_0 = e_1$, W_n just permutes it into the $e_1, \dots, e_n$ and puts some sign. Hence the orbit is

$$\{ \pm e_i \mid 1 \leq i \leq n \}.$$

By linear independence, as

$$e_1 = e_1$$
$$e_2 = e_2 - e_1 + e_1$$
$$e_3 = (e_3 - e_2) + (e_2 - e_1) + e_1$$
$$\vdots$$

we have e_i are the ones in Φ^+ and $-e_i$ the ones in Φ^-.

- For $(e_{i+1}-e_i)=\alpha_i$: H_n permutes the basis as permutations.

There exists one permutation sending $i+1 \longrightarrow x$, $i \longrightarrow y$ for any $x \neq y$. Pick such $\sigma \in H_n$ and to put signs multiply by the appropriate diagonal matrix in N_n.

$\therefore$ The orbit of $e_{i+1}-e_i$ is $\{\pm e_x \pm e_y,\ x \neq y\}$.

This gives the other part of the root system. Again the positive ones are

$$\{ e_i \pm e_j \mid 1 \leq i < j \leq n \}.$$

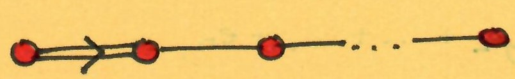
—o—

Notice we have proved that roots of the same length form orbits.

Proposition: Repeating the same construction but with

$$\alpha_0 = 2e_1, \quad \alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-1} = e_n - e_{n-1},$$

we obtain a root system of type C_n:



The roots are

$$\{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm 2e_i \mid 1 \leq i \leq n \}.$$

We also know that subgroups generated by subsets of the reflections are Coxeter Systems on their own right. Applying this to H_n and $S = \{s_1, \dots, s_{n-1}\}$ we deduce that (H_n, S) is a Coxeter System

Proposition: Let $n \geq 2$ and $H_n = \langle s_1, \dots, s_{n-1} \rangle \leq W_n$. Then the pair $(H_n, \{s_1, \dots, s_{n-1}\})$ is a Coxeter system associated with the Cartan matrix given by the Dynkin Diagram

A_{n-1}

With the above notations, the simple roots are $\alpha_1, \dots, \alpha_{n-1}$ and the positive roots are $\alpha_i + \alpha_{i+1} + \dots + \alpha_j$, $(1 \leq i \leq j \leq n-1)$. Thus there are $\frac{n(n-1)}{2}$ positive roots in a reduced root system of type A_{n-1}.

Proof:

The only thing left to check is the positive roots. We have already found that the orbit of

$$\alpha_i = e_{i+1} - e_i$$

consists of $e_x - e_y$ (now we cannot chang signs!), $x \neq y$

If $x > y$ we can rewrite this as:

$$e_x = \alpha_{x-1} + \alpha_{x-2} + \dots + \alpha_1 + e_0$$

$$e_y = \alpha_{y-1} + \alpha_{y-2} + \dots + \alpha_1 + e_0,$$

hence:

$$e_x - e_y = \alpha_{x-1} + \dots + \alpha_y, \quad 1 \leq y < x \leq n,$$

In other words,

$$\alpha_j + \alpha_{j-1} + \dots + \alpha_i, \qquad 1 \leq i \leq j \leq n-1.$$

-·-

We are only missing type D_n. We now study this case.

Definition: Considering $(W_n, \{t, s_1, \dots, s_{n-1}\})$ we can define a homomorphism

$$\varepsilon : W_n \longrightarrow \{\pm 1\}$$

defined by

$$\varepsilon(t) = -1, \quad \varepsilon(s_i) = 1.$$

Let $W_n' = \ker(\varepsilon)$.

The reason why we can construct such homomorphism is because the defining relationships all include an even number of t's, so that the corresponding map from the free group descends.

Fact: $W_n' \triangleleft W_n$ and $W_n / W_n' \cong \{\pm 1\}$ with $w \in W_n'$ iff in any presentation of w as generators, there are an even number of t's.

We will define $N_n' := W_n' \cap N_n$ and $u = t s_1 t \in W_n'$.
Notice that

$$u^2 = t s_1 t t s_1 t = t s_1 s_1 t = tt = 1$$

The matrix u is:

$$u = \left(\begin{array}{cc|c} 0 & -1 & \\ -1 & 0 & \\ \hline & & 1 \\ & & \ddots \\ & & \quad 1 \end{array}\right)$$

Furthermore, we now define

$$u_1 = u s_1, \qquad u_i = s_i u_{i-1} s_i \qquad 2 \le i \le n-1.$$

Proposition: N_n' is a subgroup of index 2 in N_n and it is generated by $u_1, \dots, u_{n-1}$. Hence, we have a semidirect product $W_n' = N_n' \rtimes H_n$.

Proof:

Notice that

$$u_1 = u s_1 = t s_1 t s_1 = t t_1.$$

Hence

$$\begin{aligned} u_2 &= s_2 u_1 s_2 \\ &= s_2 t t_1 s_2 \\ &= t s_2 t_1 s_2 \\ &= t t_2, \end{aligned}$$

and so on:

$$u_i = t t_i \qquad (*)$$

We conclude $u_i \in N_n'$, as $(*)$ shows it is diagnal, and

$$\begin{aligned} \varepsilon(u_i) &= \varepsilon(s_i u_{i-1} s_i) \\ &= \varepsilon(u_{i-1}) \\ &= \dots \\ &= \varepsilon(u_1) \\ &= \varepsilon(t s_1 t) \\ &= 1. \end{aligned}$$

Every element in N_n can be written as a product of ~~elements~~ $t_{i_1} t_{i_2} \dots t_{i_k}$, $0 \le i_1 < \dots < i_k \le n-1$, and

$$\varepsilon(t_{i_1} \dots t_{i_k}) = \varepsilon(t_{i_1}) \dots \varepsilon(t_{i_k}) = \varepsilon(t u_{i_1}) \dots \varepsilon(t u_{i_k}) = (-1)^k,$$

so only those with an even number of factors belong to N_n'.

And in such case, using that t & u_i commute as they are diagonal, and there are an even number of t's, we get they are generated by the u_i.

$\therefore$ $N_n' = \langle u_1, \ldots, u_{n-1} \rangle$ and of course it has index 2 (the other coset formed by those with odd number of t_i's).

We conclude $W_n' = N_n' \rtimes H_n$.

–o–

Hence to get generators of W_n' we only need those of H_n, which are $s_1, \ldots, s_{n-1}$ acting (by conjugation) on those of N_n' which are $u_1, \ldots, u_{n-1}$. But recall:

$$u_i = s_i u_{i-1} s_i,$$

hence, ~~a priori~~ it suffices $s_1, \ldots, s_{n-1}$ and its action on u_1. Furthermore, $u_1 = u s_1$, hence to generate W_n' it suffices to use $u, s_1, \ldots, s_{n-1}$.

Proposition: The following relations hold:

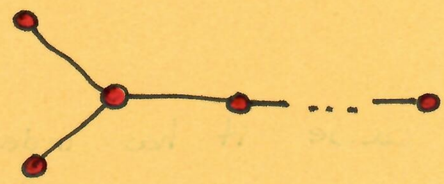
- $(us_1)^2 = 1$,
- $(us_2)^3 = 1$,
- $(us_i)^2 = 1$

which complement those we already know $(s_i s_{i+1})^3 = 1$ and $(s_i s_j)^2 = 1$ if $|i-j| > 1$.

Proof: This is just done following the definitions of u.

–o–

This relation shows this $(W_n', \{u, s_1, \ldots, s_{n-1}\})$ is a quotient of the Coxeter system given by D_n:



Theorem: Let $n \geq 4$ and $W_n' = \langle S' \rangle \subseteq GL_n(\mathbb{R})$ as above. We define $C' = (c_{ij}')_{0 \leq i,j \leq n-1}$ by

$$c_{ij}' = \frac{2(\alpha_i', \alpha_j')}{(\alpha_i', \alpha_i')},$$

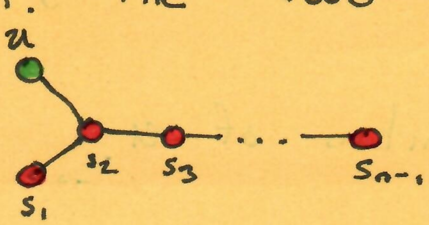
where

$$\alpha_0' = e_1 + e_2, \quad \alpha_1' := e_2 - e_1, \ \ldots, \ \alpha_{n-1}' = e_n - e_{n-1}.$$

Then C is a Cartan Matrix for the Dynkin Diagram D_n, $n \geq 4$, and $u, s_1, \ldots, s_{n-1}$ are reflections with respect to C' and with simple roots $\alpha_0', \alpha_1', \ldots, \alpha_{n-1}'$, respectively. Thus (W_n', S') is a Coxeter System of type D_n.

Moreover, the map $u \longmapsto t s_1 t$, $s_i \longmapsto s_i$ $(i \geq 1)$ defines an embedding of W_n' into W_n, where (W_n, S) is of type B_n.

We will not prove this, but I will comment on the last part. The two related Dynkin diagrams are:



W_n is generated as a Coxeter System by $\{t, s_1, \ldots, s_{n-1}\} = S$ Among these, $s_1, \ldots, s_{n-1}$ are in W_n'. Since $s_1, \ldots, s_{n-1}$ are part of S they generate part of W_n' and the question is whether they generate everything.

Notice that conjugation by t transforms S into $\{t\} \cup \{ts_1t, s_2, s_3, \ldots, s_{n-1}\}$, as t doesn't commute only with s_1. As conjugation doesn't change the value of ε, all these means that tSt is also in W_n'.

But tS_1t cannot be generated by $s_1, \ldots, s_{n-1}$ as the defining relationships forbid it.

$\therefore$ $ts_1t = u$ is a new element to be considered when looking for generators of W_n'. You only care about this one as any block that is not of the form

$$ts_1ts_1 \ldots s_1t$$

can be broken up into parts already generated by $s_1, s_2, \ldots, s_{n-1}$ (using that t commutes with $s_2, \ldots, s_{n-1}$).

Of this only possibly ts_1t could be new as the rest already are formed by ts_1t, s_1, by the defining relationships.

In a few words, ε has made the generators to branch, the only one who branches here is s_1 into

s_1 & ts_1t.