Let (W, S) be a Coxeter System. We recall the following definition:

Definition: For $J \subseteq S$ we call

$$W_J := \langle J \rangle \subseteq W$$

a standard parabolic subgroup. Its conjugates are called the parabolic subgroups.

We know that (W_J, J) is again a Coxeter System. We will be interested in the right cosets $W_J w$ of these standard parabolic subgroups.

Proposition: Let $J \subseteq S$ and define

$$X_J = \{ w \in W \mid \ell(sw) > \ell(w) \ \forall s \in J \}.$$

a) For each $w \in W$ there excists a unique $v \in W_J$ and a unique $x \in X_J$ such that $w = vx$. Moreover,

$$\ell(w) = \ell(v) + \ell(x).$$

b) For any $x \in W$ the following are equivalent:

i) $x \in X_J$,

ii) $\ell(vx) = \ell(v) + \ell(x) \quad \forall v \in W_J$,

iii) x is the unique element of minimal length in $W_J x$.

In particular, X_J is a complete set of ~~distinguished~~ right coset representatives of W_J in W.

Proof:

a) Let $w \in W$ and let x be of minimal length in $W_J w$. We can write

$$w = vx, \qquad \text{for some } v \in W_J.$$

This of course implies

$$\ell(\omega) \leq \ell(v) + \ell(x).$$

We want to prove the inequality is strict so write:

$$\ell(\omega) < \ell(v) + \ell(x)$$

Therefore there exists $v' \in W_J$ and x' ~~$\in W_J$~~ obtane by cancelling two factors in the expression vx. This is the cancellation property (C), and of course, it might happen that $v' = v$ or $x' = x$.

Notice that $v' \in W_J$ as cancelling some letter doesn't change that the generators are in J. Hence, since

$$\omega = v'x',$$

then $x' \in W_J\omega$. This implies, as x was of minimal length, that

$$\ell(x') = \ell(x).$$

Since x' is a subword of x, we have $x' = x$ and no cancellation occured there.

$\therefore$ $\ell(v') + 2 = \ell(v)$ and we have obtained

$$\omega = v'x,$$

but reducing the length of v'. We can repeat this until v' cannot be reduced more and get some expression, relabelling,

$$\omega = vx, \qquad x \in W_J\omega$$
$$v \in W_J$$

with $\ell(\omega) = \ell(v) + \ell(x)$. ⌈Notice that for each x the v is unique.⌋

We now prove that in this decomposition $x \in X_J$. Indeed take $s \in J$ and put $x' = sx$. We have

$$\ell(x') > \ell(x) \quad \text{or} \quad \ell(x') < \ell(x),$$

as the exchange condition forbids $\ell(x') = \ell(x)$. Suppose $\ell(x') < \ell(x)$ and put $\nu' = \nu s$. We then have:

$$\nu' x' = (\nu s)(sx) = \nu x = w,$$

from where $x' \in W_J w$, $\nu' \in W_J$ as $s \in J$, and so we have contradicted the minimality of x.

$\therefore \ \ell(x') > \ell(x)$.

$\therefore \ x \in X_J$.

Now we prove the uniqueness of x. This is equivalent to proving

$$W_J x \cap X_J = \{x\}$$

since from any decomposition

$$\nu_1 x_1 = \nu x$$

we get

$$x_1 = \nu_1^{-1} \nu x \in W_J x \cap X_J,$$

as x_1 would have to be in X_J as well. Pick then $y \in W_J x \cap X_J$, and repeat what we did with x instead of w, we get

$$y = \tilde{\nu} x, \qquad \tilde{\nu} \in W_J$$

and $\ell(y) = \ell(\tilde{\nu}) + \ell(x)$. If $y \neq x$ then $\tilde{\nu} \neq 1$ and so there exists $s \in J$ with

$$\ell(s\tilde{v}) < \ell(\tilde{v}).$$

But then

$$\ell(sy) = \ell(s\tilde{v}x) \le \ell(s\tilde{v}) + \ell(x) < \ell(\tilde{v}) + \ell(x) = \ell(y),$$

~~with~~ which contradicts $y \in X_J$.

$\therefore y = x$.

This concludes (a).

We now prove (b):

(i) $\Rightarrow$ (ii): If $x \in X_J$ & $v \in W_J$ then for $vx =: w$ this is the decomposition of the first part. Hence

$$\ell(vx) = \ell(v) + \ell(x).$$

(ii) $\Rightarrow$ (iii): This implies

$$\ell(vx) = \ell(v) + \ell(x) \ge \ell(x),$$

with equality only if $v = 1$.

$\therefore x$ has minimal length in the coset $W_J x$.

(iii) $\Rightarrow$ (i):

If this happens, in particular

$$\ell(sx) > \ell(x) \qquad \forall s \in J.$$

This implies of course, by definition, $x \in X_J$.

Definition: The set X_J is called the set of distinguished right coset representatives of W_J in W.

We now study an algorithm to find the right coset representatives:

Algorithm: Let (W,S) be a Coxeter system and $J \subseteq S$ any subset.

Step 1: Set $k=0$, $Y_0 = \{1\}$ and $X = Y_0$.

Step 2: Set $k = k+1$ and

$$Y_k = \{xs \mid x \in Y_{k-1},\ s \in S,\ \ell(xs) > \ell(x) \text{ and } \ell(txs) > \ell(xs)\ \forall t \in J\}$$

Set $X = X \cup Y_k$.

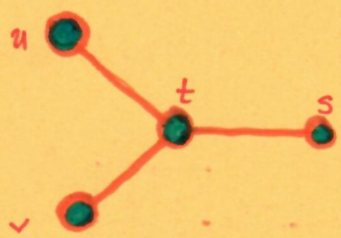
Step 3: Repeat step 2 until $Y_k = \emptyset$. Then set

$$X_J = X,$$

and stop.

Before we prove this argument works and analyze it a bit more we will see an example.

Example: Our example will be with a finite Coxeter System of type D_4. It's Dynkin diagram is



This is associated with the presentation

$$W = \langle u, v, t, s \mid (ut)^3 = (ts)^3 = (tv)^3 = (uv)^2 = (us)^2 = (vs)^2 = u^2 = v^2 = t^2 = s^2 = 1 \rangle.$$

and let us take

$$J = \{u, t\}.$$

Then, of course, W_J has the presentation

$$W_J = \langle u, t \mid u^2 = t^2 = (ut)^3 = 1 \rangle$$

which is of type A_2 and has 6 elements:

$$W_J = \{1, u, t, ut, tu, tut = utu\}.$$

On the other hand, we know $|W| = 192$. This means that

$$|W_J \backslash W| = \frac{192}{6} = 32.$$

There are 32 cosets. The idea of the algorithm is to find the representatives, from smallest in length upwards.

In X_J we always have 1 and that is how we begin

- $k = 0$ (initial length)
- $Y_0 = \{1\}$ (those elements of X_J with length $\leq k$)
- $X = \{1\}$ (those element of X_J with length $= k$)

In order to find those elements of X_J of length 1 we do:

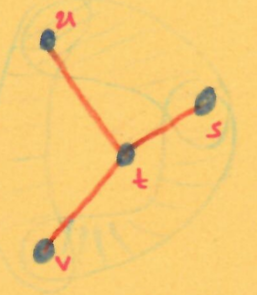
$$Y_1 = \{xp \mid \underbrace{x \in Y_0}_{x=1},\ p \in S,\ \underbrace{\ell(xp) > \ell(x)}_{\substack{\text{This always}\\ \text{holds if } p \in S,\\ x=1}} \ \&\ \ell(txp) > \ell(xp)\ \forall t \in J\}.$$

	p			xp	u	t	
	u	$1 = \ell(u)$	$\ell(1) = 0$	u	1	tu	
	v	$1 = \ell(v)$	$\ell(1) = 0$	v	uv	tv	$\longrightarrow v$
$x = 1$	s	$1 = \ell(s)$	$\ell(1) = 0$	s	us	ts	$\longrightarrow s$
	t	$1 = \ell(t)$	$\ell(1) = 0$	t	ut	1	

Hence $Y_1 = \{v, s\}$ and now $X = X_0 \cup Y_1 = \{1, v, s\}$.

Now we compute those elements of X_J with length 2.

$$Y_2 = \{xp \mid x \in Y_1,\ \ p \in S \ \ \ell(xp) > \ell(x) \ \ \& \ \ \ell(txp) > \ell(xp) \ \ \forall t \in J\}$$

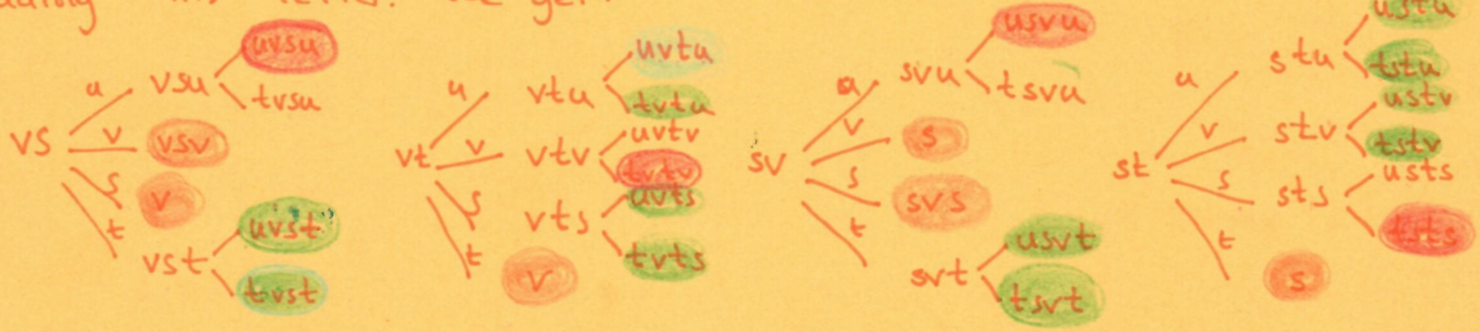


x	p	xp	ℓ(xp)	ℓ(xp) > ℓ(x)	xp	u	t	
v	u	vu	2	Y	vu	uvu	tvu	N
	v	1	0	N				
	s	vs	2	Y	vs	uvs	tvs	Y
	t	vt	2	Y	vt	uvt	tvt	Y
s	u	su	2	Y	su	usu	tsu	N
	v	sv	2	Y	sv	usv	tsv	Y
	s	1	0	N				
	t	st	2	Y	st	ust	tst	Y

Hence we conclude $Y_2 = \{vs, vt, sv, st\}$. Hence we have found so far:

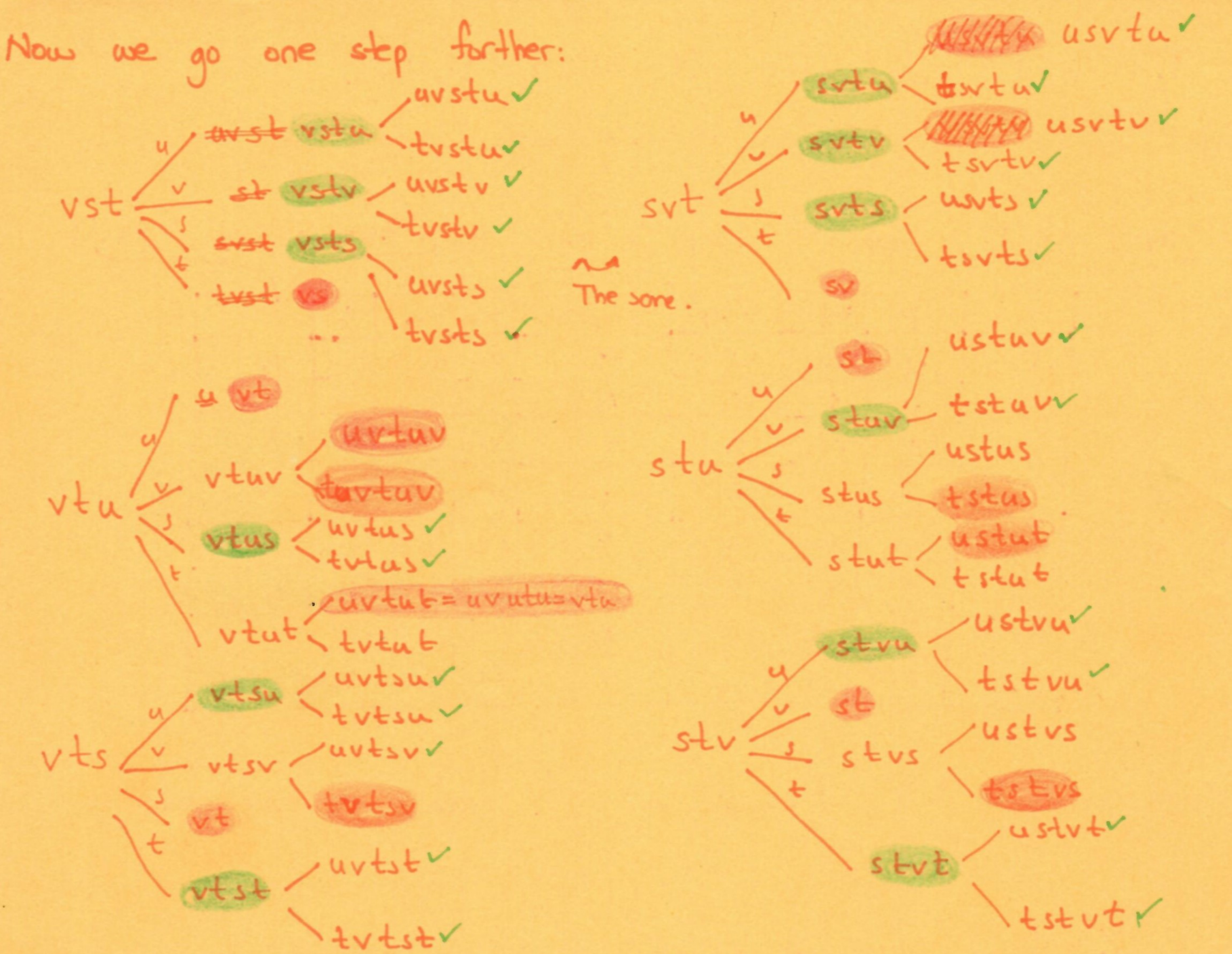
$$X = \{1, v, s, vs, vt, sv, st\}$$

For Y_3 we now understand how this works. Firstable we check every possible letter we can add to the words in Y_2 to genuinly increase length. Length must go up unless some relationship is formed by adding this letter. We get:



And for each one of this we check if either of u or t fails to increase the length when put at the front. Checking the cases and argumenting we get $Y_3 = \{vst, vtu, vts, svt, stu, stv\}$

Now we go one step further:



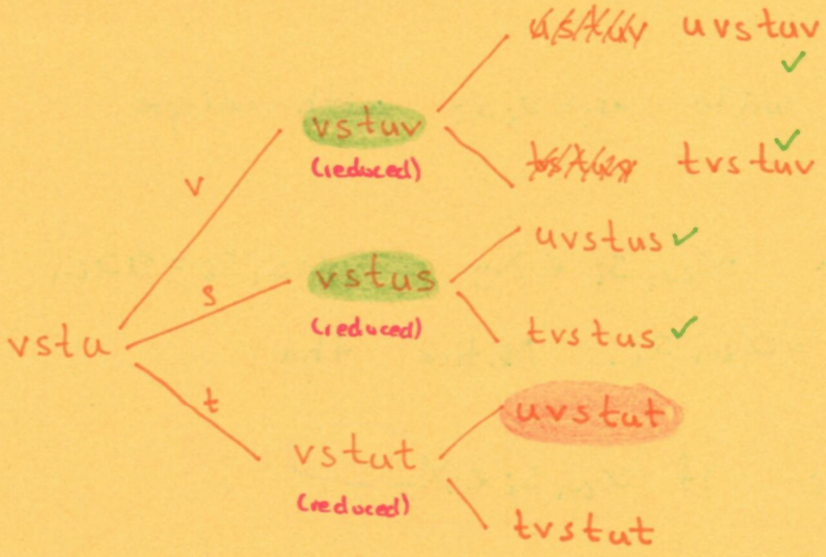
Notice that: $xytyx = xy\underset{\sim}{txy} = xy\,xtxt = x^2y\,txt = ytxt$, which is reduced. This eliminates several above. In order to do the checkings we used the folding criterion & Tit's criterion of equivalence.

Hence we have found:

$Y_4 = \{vstu, vstv, vsts, vtus, vtsu, vtst, svtu, svtv, svts, stuv,$ $stvu, stvt\}$

So far the algorithm has produces $\overset{\text{almost}}{\vee}$ $12 + 7 + 6 = 12 + 13 = 25$. We are missing $\underset{\text{at least}}{\wedge}$ 7 cosets. ~~Let's see what happens with vstu~~

In reality we have created 16 so we are half way!



When we go and check for these words to be reduced we already have a lot of reduced structures stored before..

This process adds two new ones, and using appropiate symmetries you can see what we add and eliminate repetitions.

— o —

These lemmas are fundamented in the following two lemmas:

Lemma (Deodhar's Lemma): Let $J \subseteq S$, $x \in X_J$ and $s \in S$. Then either $xs \in X_J$ or $xs = ux$ for some $u \in J$.

Lemma: Let $J \subseteq S$, $x \in X_J$ and $w \in W$ such that $xw \in X_J$. If $w = s_1 \dots s_r$ is reduced for w, with $s_j \in S$, then we have $x s_1 \dots s_i \in X_J$ for all $i \leq r$.

Proof:

① Deodhar's: If $xs \notin X_J$ then there exists $u \in J$ such that $l(uxs) < l(xs)$. We also have $l(ux) > l(x)$, since $x \in X_J$. But $l(uxs) \pm 1 = l(ux) > l(x)$, which implies

$$l(x) < l(xs),$$

i.e. $l(xs) = l(x) + 1$. As $l(ux) = l(x) + 1$, we deduce by folding

Proof of lemma:

Let $y_i = x s_1 \dots s_i$ and we write $y_i = v_i x_i$ with unique $v_i \in W_J$ & $x_i \in X_J$.

Then by Deodhar's lemma: $x_{i-1} s_i \in X_J$ or $x_{i-1} s_i = u x_{i-1}$ for some $u \in J$. Writing $t_i = x_{i-1}^{-1} x_i$. Notice that

$$t_i = \begin{cases} s_i & \text{if } x_{i-1} s_i \in X_J \\ 1 & \text{otherwise} \end{cases}$$

But then

$$x t_1 \dots t_r = x w$$

and $t_1 \dots t_r = w$. Since $\ell(w) = r$ we have $t_i = s_i$ always.

$\therefore\ x_i \in X_J$ for all i.

Now suppose $J \subseteq K \subseteq S$ are ~~parab~~ subsets and consider the parabolic subgroups:

$$W_J \subseteq W_K \subseteq W_S.$$

We can write X_J^K for the distinguished right coset reps. of W_J in W_K.

Notice that:

$$W_J \cdot X_J = W = W_K \cdot X_K = (W_J \cdot X_J^K) \cdot X_K$$

Notice then that for every ω we can write:

$$\omega_j' \cdot x_J = \omega_j \cdot x_j^k \cdot x_k$$

but

$$l(\nu x_j^k x_k) = l(\nu x_j^k) + l(x_k) \qquad \forall \nu \in W_J \text{ as then } \nu x_j^k \in W_K.$$

$$= l(\nu) + l(x_j^k) + l(x_k) \qquad \text{since } x_j^k \in X_J^k \text{ and length is independent of parabolic } \cancel{\text{subset}}\text{. subgroup.}$$

$$\neq$$

$$= l(\nu) + l(x_j^k x_k) \qquad \text{as } x_j^k \in W_K$$

$\therefore x_j^k x_k \in X_K$. By uniqueness of decomposition:

$$x_J = x_j^k x_k ;$$

which we write:

$$\boxed{X_J = X_J^k \cdot X_K.}$$

Remark: Notice that we can repeat this argument with left cosets and we get similar results. In particular, the inverse takes X_J into X_J^{-1} which is a set of distinguished left coset reps. of minimal length as $l(\omega) = l(\omega^{-1})$.

Theorem: Let $J, K \subseteq S$ and define $X_{JK} = X_J \cap X_K^{-1}$. Then for each $w \in W$ there are $u \in W_J$, $v \in W_K$ and a unique $d \in X_{JK}$ such that $w = udv$ and $\ell(w) = \ell(u) + \ell(d) + \ell(v)$.

Moreover, for each $d \in W$, the following are equivalent:

i) $d \in X_{JK}$

ii) d is the unique element of minimal length in the double coset $W_J d W_K$.

In particular, X_{JK} is a complete set of double coset reps. of $W_J d W_K$ in W.

We will not prove this result. We instead want to study more carefully the set X_J.

→ From now on suppose (W, S) is spherical.

Lemma: Let w_0 be the longest element of W, w_J the longest element of W_J and define $d_J = w_J w_0$. Then

i) d_J is the unique element of maximal length in X_J.

ii) $X_J = \{ w \in W \mid w \text{ is a prefix of } d_J \}$.

iii) $d_J^{-1} = d_J^{w_0} = d_{J'}$ where $J' = J^{w_0}$.

Remark: i) A prefix of a word w is any element x such that

$$\ell(x^{-1}w) = \ell(w) - \ell(x).$$

In general what happens is $\ell(x^{-1}w) \geq \ell(w) - \ell(x)$.

In this case there is $y \in W$ s.t. $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. Basically then, prefixes are words that appear at the beginning of reduced expressions of w.

ii) The longest element is unique and satisfies

$$\cancel{\ell(w_0 w) = \ell(w) - \ell(w_0)}$$

$$\boxed{\ell(w w_0) = \ell(w_0) - \ell(w)}$$

In particular, for $s \in S$ we have:

$$\begin{aligned} \ell(w_0 s w_0) &= \ell(w_0) - \ell(w_0 s) \\ &= \ell(w_0) - \ell(s w_0) \qquad \text{as } s^2 = w_0^2 = 1 \\ &= \ell(w_0) - (\ell(w_0) - \ell(s)) \\ &= \ell(s). \end{aligned}$$

$\therefore\ w_0 S w_0 = S.$

Proof:

a) Let $x \in X_J$ of maximal length. Then, since $W = W_J \cdot X_J$ we get

$$\ell(w) = \ell(w' \cdot x') = \ell(w') + \ell(x') \leq \ell(w_J) + \ell(x) = \ell(w_J x).$$

$\therefore\ w_J x$ has maximal length.

$\therefore\ w_J x = w_0.$

b) Let $x \in X_J$. By Deodhar's lemma $xs \in X_J$ whenever $\ell(xs) < \ell(x)$ (by folding basically). Hence, if x is written in reduced form, applying this we get (cancelling terms from right to left) that the prefix is in X_J.

In particular, every prefix of d_J is in X_J.

Now we need to prove the converse. Suppose first that $x \in X_J$ is such that $\forall s \in S$ ~~and~~ $xs \in X_J$ and $l(xs) < l(x)$.

By Deodhar, we have $xs = ux$ for some $u \in J$. Hence:

$$l(\omega_J u) < l(\omega_J)$$

Which implies

$$\begin{aligned} l(\omega_J xs) &= l(\omega_J u) + l(x) \\ &= l(\omega_J) - 1 + l(x) \\ &= l(\omega_J) + l(xs) \\ &= l(\omega_J xs) \\ &< l(\omega_J x) \qquad \forall s. \end{aligned}$$

$\therefore \omega_J x = \omega_0$

Otherwise, if x is such that $l(xs) > l(x)$ and $xs \in X_J$ then x is a prefix of d_J by ~~in~~ downward induction.

(c) $d_J^{-1} = \omega_0 \omega_J = \omega_J^{\omega_0} \omega_0 = \omega_{J'} \omega_0 = d_{J'}$.

-o-

Definition: The coset graph Γ_J for $J \subseteq S$ is the labelled directed graph with vertex set X_J where, for a vertex $x \in X_J$ and any $s \in S$ we draw an edge with label $s \in S$ from x to xs if $xs \in X_J$ and $l(xs) > l(x)$. We also write $x \xrightarrow{s} xs$ for this edge.