During this lecture we assume (W, S) is spherical.

Definition: A conjugacy class C of W is called cuspidal class if

$$C \cap W_J = \emptyset \qquad \forall J \subsetneq S.$$

That is, if C does not intersect any proper parabolic subgroup. Furthermore, for every conjugacy class C of W we denote

$$C_{min} = \{ w \in W \mid \ell(w) \leq \ell(v) \quad \forall v \in C \}.$$

we begin our study with a particular class of elements that are relevant for us and that have already appeared before:

Definition: A Coxeter Element of (W, S) is a product of all $s \in S$ in any given order

Theorem [Shi 1997]: Let (W, S) be any coxeter system. If e is the number of edges of the Coxeter graph of W, the number of Coxeter elements is 2^e.

Proof: (Recall we assume W is spherical!)

The idea first is to realize that we can deal with irreducible Coxeter diagrams, since coxeter elements always break up as product of those of different components.

Once in an irreducible one, you write down following some order the letters and you chose whether to write it ~~left~~ or right of those letters it doesn't commute. This gives the 2^e choices. It is well defined since there are no cycles in these diagrams.

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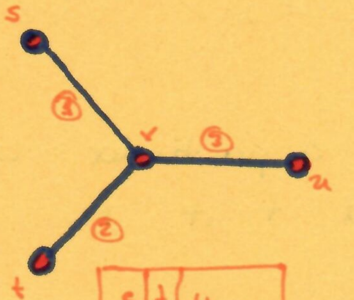
<u>Example</u>: We will do it for D_4:



We will begin with pivot v and read R or L in order 1, 2, 3.

s	t	u	
R	R	R	vstu
R	R	L	uvst
R	L	R	tvsu
L	R	R	svtu

s	t	u	
R	L	L	tuvs
L	R	L	usvt
L	L	R	stvu
L	L	L	stuv

All of these coxeter elements are different as there are no M-operations that can be performed on them.

∴ We find the 8 coxeter elements!

<u>Definition</u>: We say w and $w' \in W$ are conjugate by cyclic shift if there exists a sequence

$$w = v_0,\ v_1, \dots, v_m = w'$$

of elements of W and elements $x_i, y_i \in W$ such that

$$v_{i-1} = x_i y_i, \qquad v_i = y_i x_i$$

and

$$\ell(v_{i-1}) = \ell(x_i) + \ell(y_i) = \ell(v_i)$$

for all $i = 1, \dots, m$.

Notice that this implies in particular:

$$v_{i-1} = x_i v_i x_i^{-1},$$

hence w is conjugate to w'.

Theorem: Any two Coxeter elements of (W, S) are conjugate by cyclic shift.

Proof:

This is trivial if $|S| \leq 1$. Suppose then $|S| \geq 2$ and write

$$S = \{s_1, \dots, s_n\}, \qquad n = |S|.$$

We further suppose s_n is connected in the coxeter graph to at most one other generator. ~~xx~~ ~~xxx~~

$\therefore$ s_n does not commute with at most one other generator.

Let w be an arbitrary Coxeter element of W and write a reduced expression of it:

$$w = a s_n b.$$

Here a is the product of the elements to the left in the reduced expression and b those in the right.

By our choice of s_n it commutes with one of a or b, so that actually $w = s_n a b$ or $w_n = a b s_n$.

Clearly $s_n ab$ and $ab s_n$ are conjugate by cyclic shift (put $x = ab$, $y = s_n$ and since no letter repeats $\ell(s_n) + \ell(ab) = \ell(s_n ab)$)

Considering $J = S - \{s_n\}$, we get that ab is a coxeter element of W_J. By induction on $|S|$, then, we get ab is conjugate by cyclic shift to $s_1 \cdots s_{n-1}$. That is, there exists $v_0, v_1, \ldots, v_m$ elements of W_J and $x_i, y_i \in W_J$ with

$$ab = v_0, \quad s_1 \cdots s_{n-1} = v_m$$

and

$$v_{i-1} = x_i y_i, \quad v_i = y_i x_i,$$

$$\ell(v_{i-1}) = \ell(x_i) + \ell(y_i) = \ell(v_i).$$

Now we claim $v_0, \ldots, v_m$ are all coxeter elements of W_J. Indeed this follows from the fact that the first one and at each time we are just permuting the elements. In particular all letters appear and no one repeats.

$\therefore w$ & $s_1 \dots s_n$ are conjugate by cyclic shift if we can show

$$x_i y_i s_n \quad \& \quad y_i x_i s_n$$

are conjugate by cyclic shift.

Fix an index $i \in \{1, \dots, m\}$. Since $x_i y_i s_n$ is a coxeter element of W where every $s_i \in S$ occurs exactly once as factor we have that $x_i y_i s_n$ & $y_i s_n x_i$ are conjugate by cyclic shift. Since s_n commutes with x_i or y_i ~~wb~~ we have

$$x_i y_i s_n = x_i s_n y_i \quad \text{or} \quad y_i s_n x_i = s_n y_i x_i.$$

Clearly $s_n y_i x_i$ and $y_i x_i s_n$ are conjugate by cyclic shift.

We are done. -o-

Lemma: Let $J \subseteq S$ and let w_c be a Coxeter element of W. Then w_c fixes a point in the action of W on the cosets of W_J if and only if $J = S$.

Proof:

Recall we are writing the action by

$$(W_x) \cdot \omega = W_{x\omega}.$$

Let W_J be a given coset and X_J the set of representatives, consisting of those elements of minimal length in their coset. What we need to prove is that

$$W_J x w_c = W_J x \iff J = S.$$

If $J = S$ then $W_J = W$ and this is obvious. Now we suppose

$$W_J x w_c = W_J x$$

for some $x \in X_J$, and write $w_c = s_1 \circ \dots \circ s_n$. We will build several elements $x_0, \dots, x_n$, $u_1, \dots u_n$ as follows:

$$(*) \quad x_0 = x, \qquad \left.\begin{aligned} x_0 s_1 &= u_1 x_1 \\ x_1 s_2 &= u_2 x_2 \\ x_2 s_3 &= u_3 x_3 \\ &\vdots \\ x_{n-1} s_n &= u_n x_n \end{aligned}\right\} \text{ where we can do this, inductively as, } W = W_J \circ X_J,$$

and here $x_0, \dots, x_n \in X_J$, $u_1, \dots, u_n \in W_J$. We then have:

$$\begin{aligned} x w_c &= x_0 s_1 \dots s_n \\ &= u_1 x_1 s_2 \dots s_n \\ &= u_1 u_2 x_2 \dots s_n \\ &= u_1 \dots u_n x_n. \end{aligned}$$

At each step of the construction $(*)$ we have, as $x_j \in X_J$ and $s_i \in S$, that $x_{i-1} s_i = x_i$ & $u_i = 1$ or $u_i \in J$, $x_{i-1} = x_i$. This is the content of Deodhar's lemma.

$\therefore$ we set $t_i = 1$ or $t_i = s_i$ according to case such that

$$\boxed{x_i = x_{i-1} t_i}$$

$\therefore$ $\{t_1, \dots, t_n\} \subseteq S$ and $t_1 \dots t_n$ is a coxeter element of some parabolic subgroup W_K of W and

$$\boxed{x_0 t_1 \cdots t_n = x_n.} \qquad (\Delta)$$

Recall our hypothesis $W_J x w_c = W_J x$, that is, $x w_c$ is represented by x_0. But $x w_c = u_1 \cdots u_n x_n$ and each $u_i = 1$ or a member of J, so $u_1 \cdots u_n \in W_J$.

$\therefore$ $x w_c \in W_J x_n$.

$\therefore$ $x w_c$ is represented by x_n. Hence, by (Δ), we have

$$t_1 \cdots t_n = 1,$$

as we can cancel $x_0 = x = x_n$.

The only parabolic subgroup with 1 as a Coxeter element is W_K, with $K = \emptyset$.

$\therefore$ $t_i = 1$ $\forall i$ (recall each t_i was an s_i and these appeared in order $s_1, \ldots, s_n$ so there is no repetition).

$\therefore$ $x_i = x$ $\forall i$, and so $u_i = x s_i x^{-1}$ with $u_i \in S$.

$\therefore$ $J = x S x^{-1} = S$. (since $u_i \in J \subseteq S$ and $x \cdot x^{-1}$ is a conjugation so it is bijective.)

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Theorem: ~~Let C be the special case of Coxeter element~~
Let C be the ~~special~~ conjugacy class of W that contains the Coxeter Elements.

i) C is a Cuspidal Class.

ii) $C_{min} = \{ w \mid w \text{ is a coxeter element of } W \}$.

Proof:

i) If $C \cap W_J \neq \emptyset$ pick $x \in C \cap W_J$ and write $y w_c y^{-1} = x$ for some Coxeter Element w_c.

Then

$$(W_J y) \cdot w_c = W_J y w_c = W_J x y = W_J y,$$

which means w_c fixes a coset of W_J.

$\therefore J = S$, i.e $W_J = W$.

$\therefore$ C is cuspidal.

ii) If $y \in C$ is such that $\ell(y) < n$, then by force (by using a reduced expression) some letter is missing and so $y \in W_J$ for some $J \subsetneq S$.

$\therefore$ C wouldn't be cuspidal.

$\therefore$ $\ell(y) \geq n$

If $\ell(y) = n$, then in a reduced expression all letters must appear since otherwise the same argument applies.

$\therefore$ y is a Coxeter element, as desired.

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We now study more general cuspidal classes.

Definition: For each $J \subseteq S$ we define a map

$$\begin{aligned} \pi_J : W &\longrightarrow \mathbb{N}_0 \\ w &\longmapsto \pi_J(w) \end{aligned}$$

defined by

$$\pi_J(w) = \left| \mathrm{Fix}_w \left(W / W_J \right) \right|$$

Notice that from Tits reducibility criterion, for any $w \in W$ and any reduced expression we must have the same letters appearing, maybe not the same number of times. We call this set $J(w)$.

Finally define

$$\beta_{JK} = \left| \mathrm{Fix}_{W_K}\left(W/W_J \right) \right|.$$

We will not prove the next result, which is easy, just for the sake of time.

Proposition: Let $w \in W$ and C be a conjugacy class of w in W. Let $w' \in C_{min}$ and $K = J(w')$. Then

$$\pi_J(w) = \beta_{JK} \qquad \forall\, J \subseteq S.$$

In particular,

$$\pi_J(c_K) = \beta_{JK}$$

if c_K is a coxeter element of W_K, $K \subseteq S$.

We use this because it implies a characterization of cuspidal classes:

Corollary: Let $w \in W$. Then the conjugacy class C of w in W is cuspidal if and only if $\pi_J(w) = 0$ for all proper subsets $J \subset S$.

Proof:

If $x w x^{-1} \in W_J$ then w fixes the coset $W_J x$, so that $\pi_J(w) \neq 0$. That is, $\pi_J(w) = 0$ means $C \cap W_J = \varnothing$. Varying $J \subsetneq S$

$$\begin{aligned} l(w') + l(x) &= l(w'x) \quad \text{as } w' \in W_J \text{ \& } x \in X_J \\ &= l(x\, x^{-1} w' x) \\ &= l(xw) \\ &\leq l(x) + l(w). \end{aligned}$$

Hence, $l(w') \leq l(w)$, but $w \in C_{min}$. Hence $w' \in C_{min}$ too.

$\therefore$ As $J(w)$ & $J(w')$ are conjugate we get $J(w') = S$, i.e. $J = S$ as otherwise w' could be written with less letters.

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Proposition: Let $J, K \subseteq S$ and let c_J & c_K be Coxeter elements of W_J, W_K, respectively. Then c_J & c_K are conjugate in W if and only if J and K are conjugate in W.

Proof:

- If J and K are conjugate in W then c_J is conjugate to some coxeter c_K' in ~~W_J~~ W_K which in turn is conjugate to c_K.

- If c_J & c_K are conjugate then they belong to the same conjugacy class in W. We claim they still have minimal length here (they do in W_J & W_K). Indeed, let $u \in W$ be such that

ii) $J(w) = S$ for all $w \in C_{min}$.

iii) There exists an element $w \in C_{min}$ such that $J(w) = S$.

Proof:

<u>(i) $\Rightarrow$ (ii)</u>: This follows from the fact that if C is cuspidal then $J(w) = S$ $\forall w \in C$, since otherwise $C \cap W_{J(w)} \neq \emptyset$ & $J(w) \subsetneq S$.

<u>(ii) $\Rightarrow$ (iii)</u>: Obvious.

<u>(iii) $\Rightarrow$ (i)</u>: Let $w' \in C \cap W_J$ for a subset $J \subseteq S$. and pick an element $u \in W$ such that

$$u^{-1} w' u = w.$$

Let us write $u = v x$ using the decomposition

$$W = W_J \cdot X_J.$$

Then we get:

$$\begin{aligned} W_J \ni v^{-1} w' v &= (u x^{-1})^{-1} w' (u x^{-1}) \\ &= x u^{-1} w' u x^{-1} \\ &= x w x^{-1} \\ &\in C. \end{aligned}$$

$\therefore$ $v^{-1} w' v \in C \cap W_J$. Hence we can assume $v = 1$, and $u \in X_J$ directly.

Now let us compute lengths:

$$\cancel{\ell(w) = \ell(x^{-1} w' x) \leq \ell(\underset{\in X_J}{x}) + \ell(\underbrace{x^{-1} w' x}_{\in W_J}),}$$

we obtain cuspidality.

Conversely, if it is cuspidal, then $J(w) = J(w') = S$ for every element w' of minimal length in the conjugacy class of w. Furthermore, for a coxeter element we also have $J(w_c) = S$.

$\therefore\ \pi_J(w) = \pi_J(w_c)$, &

and we know $\pi_J(w_c) = 0 \quad \forall$ proper $J \subseteq S$.

$\therefore$ C is cuspidal.

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Theorem (Howlett): Let C be a conjugacy class of W and let $w, w' \in C_{min}$. Then $J(w)$ is conjugate to $J(w')$.

Proof:

Let $K = J(w)$ and $K' = J(w')$. Then we have

$$\beta_{JK} = \beta_{JK'} \qquad \forall\, J \subseteq S$$

And this implies the result. (We have not seen why, this is because of the theory of Burnside rings...).

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Theorem: Let C be a conjugacy class of W. Then the following are equivalent:

i) C is a cuspidal class.

c_K^u has potential smaller length. Writing $u = vx$ with $v \in W_K$, $x \in X_K$ we get:

i) $c_K^v := v^{-1} c_K v$ still ~~was~~ is in W_K, and so $\ell(c_K) \leq \ell(c_K^v)$.

ii) On the other hand repeating the argument of previous theorem:

$$\ell(c_K^v) \leq \ell(c_K^{vx}) = \ell(c_K^u).$$

$\therefore$ $\ell(c_K) \leq \ell(c_K^u)$ for all u, and so c_K still has minimal length. Analogously for J.

$\therefore$ By Howletts result $J(c_K)$ is conjugate to $J(c_J)$.

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