University of Toronto

Classification in Mathematics

Due: April 25, 2020

> Try to solve the most you can and explain the solution to the best of your abilities. Do not just put computations, be mindful of presentations and of the quality of your writing.
> Please upload the solutions with the right orientation, since reading rotated images is very tiring and sometimes complicated.
> Make sure your problems are in order.
> These problems might be harder for some of you than others, I do encourage you to talk among yourselves if that's possible and to ask the TA or me for suggestions and help. If you get frustrated, GOOD, is part of progress. Don't let it dominate you, look for help, think, take your time. Math is worth the challenge.

Problem 1: Polynomials A single variable polynomial is an expression of the form $P(x) = a_n x^n + ... + a_1 x + a_0$. For example, $x + 3$ or $3x^2 - 7x + 1$ are both polynomials. A very classical result states that the solution to a quadratic polynomial $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. Solve the quadratic polynomial $ax^2 + c = 0$.
2. Verify that we can change of variable, by finding $y = x + p$ for some p, in such a way that the polynomial $ax^2 + bx + c = 0$ becomes $y^2 + d = 0$. This kind of change of variable is called a shifting.
3. Combine the previous two questions to give a proof of the quadratic formula. In a certain sense, what we have said is the following: firstable, we have proved that all quadratic polynomials are equivalent under shifting to one of the very simple form $x^2 + c = 0$. Then, we have verified that it is enough to solve this simple ones, since knowing its solutions gives the answer for the general case.
4. Suppose we want to find the cubic formula. Verify that there is a change of variable that transform any cubic polynomial into one of the form $x^3 + px + q = 0$.
5. If we want to repeat our success with the quadratic formula, we also need to eliminate the part px. Let us substitute $x = u + v$, and we would get:

   $$(u + v)^3 + p(u + v) + q = 0.$$

   Verify that these leads to $u^3 + v^3 + (3uv + p)(u + v) + q = 0$. Hence, if we impose the condition $u^3 + v^3 = -q$ and $3uv + p = 0$ then we find a solution $x = u + v$ to our original problem.
6. Suppose that $q^2 + 4p^3/27 > 0$. Then solve the system $u^3 + v^3 = -q$ and $3uv + p = 0$. Notice that this system can be solved by the quadratic formula, if you do it correctly, so we have gone from degree three to degree two, which we already know how to solve.
7. Verify that a solution to $x^3 + px + q$ if $q^2 + 4p^3/27 > 0$ is

   $$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Where did we use the condition $q^2 + 4p^3/27 > 0$. What changes if we do not assume that?

The exposition we did here for the cubic formula is known as Cardano's Trick or Cardano's Method. The Italian school was very much interested in solving polynomial equations, as public challenges happened as way of entretainment where one challenger would pose euqations to its rival and viceversa, and the one who could solve them faster and better would win. Understanding these formulas lead to a lot of discoveries in mathematics in several directions. The similar process for formulas of degree 4 lead to much controversies, and are known now as Ferrari's method. The situation got impossible for fifth formulas or higher, until Evariste Galois introduced his theory to explain why such succeses were not easily found anymore.

Problem 2: Similar Matrices A 2×2 matrix is a 2×2 array of numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are numbers. We can add and multiply matrices as follows:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix},$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

Finally, we can also "multiply" any matrix by any number by

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

A particularly important matrix is called the **identity matrix**

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1. Let A be any matrix. Verify that $AI = IA = A$.
2. Verify, by finding an example, that product of matrices is not commutative.
3. A matrix A is called **invertible** if there exists a matrix B such that $AB = BA = I$. Not all matrices are invertible. In this case we call B the inverse of A and denote it by A^{-1}. Find an example of such a non invertible matrix.
4. Associated to any matrix, there are two extremely important numbers called its **trace** and the **determinant**. They are defined by:

   $$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

   and

   $$\mathrm{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

   Prove that $\det(AB) = \det(A)\det(B)$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
5. Prove that a matrix is invertible if and only if its determinant is different of 0. Find a formula for its inverse in terms of the entries. (If you need help, please talk to us).
6. Two invertible matrices X and Y are called **conjugate** if there exists an invertible matrix A such that $AXA^{-1} = Y$. Prove that the determinant and the trace are invariant under conjugation, that is, two conjugate matrices have the same determinant and the same trace.

7. The following matrices $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ have the same determinant and the same trace. Prove they are conjugate.
8. In general, determining whether the two given matrices are conjugate or not by finding the conjugating matrix can be tough. Luckily there exists the following result: *two invertible matrices are conjugate if and only if they have the same determinant and the same trace.* Try to prove this.
9. The previous result gives a classification of matrices according to its trace and its determinant. The sets of invertible matrices that are conjugate among themselves is called a conjugacy class.
10. (Challenge for you.) How do you have to modify all this, if we use 3×3 matrices?

Problem 3: Congruent Matrices The transpose of a 2×2 matrix is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

1. Verify that $\det(A^T) = \det(A)$ and $\mathrm{Tr}(A^T) = \mathrm{A}$.
2. Two matrices X and Y are called congruent if there exists an invertible matrix P such that $P^T XP = Y$. Does the determinant or trace remain invariant under congruence relationship? That is, is it true that two congruent matrices have the same determinant or trace?
3. In the previous problem we saw that determinant and trace classify conjugacy relationship in the invertible matrices, but this is not the case for congruence relationship. Can you find a nonconstant polynomial that is invariant under congruence relationship?
4. Prove that the sign of the determinant (by sign we mean: 1 if positive , -1 if negative or 0) is invariant under congruence relationship. Is it enough to classify the congruence relationship? (If you know complex numbers, assume we are considering here only real entries).
5. Can you find a set of numbers, similar to finding the determinant and trace for conjugacy relationship, that classifies the congruence relationship?

Problem 4: Partial Orderings Let S be any nonempty set. A partial order in the set S is a binary relationship $\leq$ such that it satisfies:

- $a \leq a$ (reflexivity)
- if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry)
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

For example, if the set is $\{1, 2, ..., 10\}$ we can use our usual notion of "smallness" to define $a \leq b$ if a is smaller than b in the usual sense. In the same set, we could now say $a \leq b$ if a divides b. In this new definition, 7 is not comparable to 10, for example.

1. For the set $\{a, b, c, d\}$, find all possible binary relationships that make this into a partially ordered set. (You must find 19).
2. Two partially orders in the set S are isomorphic if there is a function $f : S \to S$ (a permutation of the letters, if you will) that preserves all the binary relationships. That is, $f(a) \leq f(b)$ if and only if $a \leq b$. How many different **nonisomorphic** partially ordered sets there are for $\{a, b, c, d\}$? Notice how the question of finding all possible partial orders and all possible partial orders up to isomorphism give very different answers.
3. For any positive integer n consider is set of divisors $S(n)$. Put a partial order in $S(n)$ by declaring $a \leq b$ if a divides b. For example, for $n = 6$ we have $S(6) = \{1, 2, 3, 6\}$ and the relationships there are $1 \leq 2 \leq 6$, $1 \leq 3 \leq 6$ and $a \leq a$ for $a = 1, 2, 3, 6$. For the different nonisomorphic partial orders you found in the previous question, can you find an integer n with 4 divisors such that $S(n)$ belong to that class of partial orders?

4. Does there exists a finite set S and a partial order $\leq$ such that there is NO integer n such that $S(n)$ has that type of partial order? Another way yo put this question: does all partial orders of finite sets appear as the partial order in the divisor of some integer? Why do you think its important to know the answer to this question?