University of Toronto

Classification in Mathematics

Due: May 9, 2020

Try to solve the most you can and explain the solution to the best of your abilities. Do not just put computations, be mindful of presentations and of the quality of your writing.
Please upload the solutions with the right orientation, since reading rotated images is very tiring and sometimes complicated.
Make sure your problems are in order.
These problems might be harder for some of you than others, I do encourage you to talk among yourselves if that's possible and to ask the TA or me for suggestions and help. If you get frustrated, GOOD, is part of progress. Don't let it dominate you, look for help, think, take your time. Math is worth the challenge.
For this homework we do expect everybody to submit the homework on time and with complete solutions for the 3 problems.

1 Concepts

During the course we gave the following definition:

A **partially order set**, short written as **POSET**, is a set P with a binary relationship $\leq$ such that it satisfies:

Reflexivity: $a \leq a$ for all elements of P.

Antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$.

Transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$.

We write $a < b$ to mean $a \leq b$ but $a \neq b$.

We will focus only on POSETS that have a finite number of elements. In this situation we have the following definintion, also discussed in the class.

We say two POSETs, $(P, \leq_P)$ and $(Q, \leq_Q)$, are **isomorphic** if there exists a one-to-one and onto map $f : P \to Q$ (we called this a **relabeling** during the lecture) such that

$$a \leq_P b \text{ if and only if } f(a) \leq_Q f(b).$$

During class we defined three different families of finite POSETS that we review now:

Given a positive integer n, we denote by $[\![n]\!]$ the POSET with $P = \{1, 2, ..., n\}$ and with the usual ordering $a \leq b$ if a is smaller or equal than b in the usual sense.

Given a positive integer n, we denote by D_n the POSET with $P = \{\text{divisors of } n\}$ and with $a \leq b$ meaning a divides b. Any POSET isomorphic to one of this is called a divisor POSET.

Given a positive integer n, we denote by P_n the POSET with $P = \{\text{partition of } n\}$ and with $a \leq b$ meaning that the partition a can be taken to the partition b by a series of steps each one of which consists of adding two consecutive terms of the partition, and we require to obtain valid partitions (that is, the terms should go from bigger to smaller).

For example, for $n = 4$ we have the partition $2 + 1 + 1$ and 4. We know that $2 + 1 + 1 \leq 4$ (the $\leq$ here is defined as above, do not confuse it with the usual sense of understanding $\leq$) because we have the series of steps $2 + 1 + 1 \mapsto 3 + 1 \mapsto 4$.

2 Problems

1. The Hasse Diagram

   During the lecture we made some drawings that represent the structure of the relationship $\leq$. In this problem we will give the precise rules to create these drawings.

   (a) We will use for this problem as case study the POSET that we denote as P_6. Find all partitions of 6 (you should find 11).

   (b) In order to specify what *segments* should appear connecting elements we need the following new definition:

   Let $(P, \leq)$ be a POSET and a, b two elements of P. We say b **covers** a if $a < b$ and there is no element c such that $a < c < b$.

   Write down all cover relationships for the POSET P_6 .

   (c) We are now ready to define the drawings we saw during the lecture in more careful way:

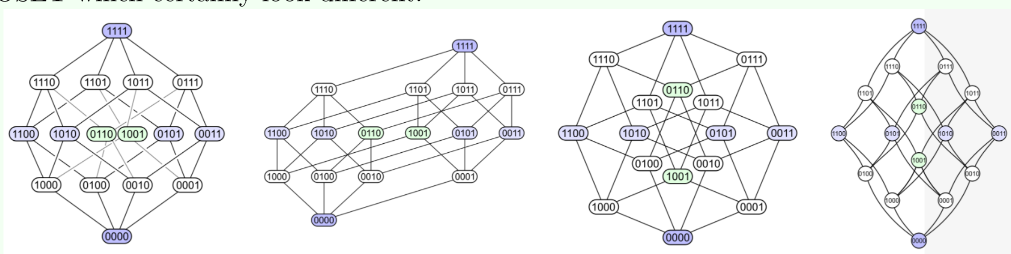
   Let $(P, \leq)$ be a POSET. A **Hasse diagram** for for this POSET is a drawing of points and lines joining some of those points following the next rules:

   1. There is a point for every element of P.
   2. If $a < b$ the the point of a appears further to the left than the point of b. We don't put restriction about who appears higher.
   3. If $a < b$ then the point of a is joined with a segment to that of b if and only if b covers a.

   Be careful! Sometimes in the definition of Hasse Diagram people change *further to the left* for *further down*.

There can be several Hasse Diagrams for the same POSET, the differences are in how they look due to the freedom of moving the height at which different points appear, and the distance between different joined points. This can help to show hidden symmetries among the diagram.

For example, taken from Wikipedia, here are four (vertical) Hasse Diagrams of the same POSET which certainly look different.



Draw two Hasse Diagram for P_6 that "highlight" different structure in it.

(d) Explain why isomorphic POSETS can be represented by the same Hasse diagram.

(e) Prove that POSETS that are represented by the same Hasse Diagram are isomorphic.

(f) Just from looking at a Hasse Diagram, how do we know if two elements are incomparable?

2. Families of POSETS

(a) Draw the Hasse Diagram of $[\![n]\!]$ for n from 1 to 5. Notice that it is always a broken line connecting points. Explain why this is always the case for $[\![n]\!]$.

(b) A special type of POSET is the following:

A POSET is called a **Total Order** or **Linear Order** if any two elements can be compared, that is, for any a and b we have $a \leq b$ or $b \leq a$.

Prove that a poset $(P, \leq)$ is a total order if and only if it is isomorphic to $[\![n]\!]$ for some n.

(c) Draw all Hasse Diagrams of D_n for $1 \leq n \leq 20$.

(d) Without drawing more Hasse Diagrams answer the following question and justify your answer: which is the smallest integer n bigger than 20 such that the Hasse Diagram of D_n is not one of those you already drew in the previous part?

(e) For which n is D_n a total order? Another way to put this question is: what are the intersections between the orders in the family $[\![n]\!]$ and D_n.

(f) Draw the Hasse Diagram of P_n for n from 1 to 5.

(g) Suppose $(P, \leq)$ is a POSET that appears in the family of divisor POSETS D_n and also appears in the partition POSETS P_n. Prove that it is a total order.

Another way to put this statement is: the redundancy (i.e. repetitions) within the divisor and partition POSETS are the total orders.

Let us elaborate a bit on the relevance of this exercise. Remember that the reason we create families is to construct in a *natural* way POSETS and hope that as we go and look into them, all possible POSETS appear at some point, that is, these families are our starting point in our search for all POSETS!

The problems that can occur as we start to manage too much information is that of repeating or missing POSETS, so it is relevant that we understand how these mistakes can occur.

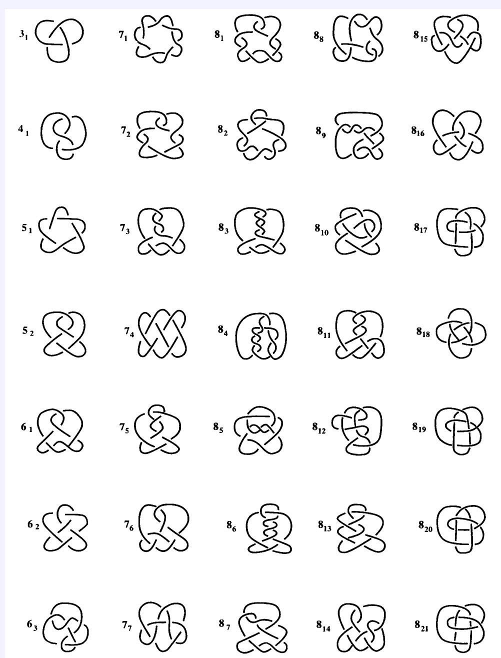
In our case, then, if we are listing POSETS we know those that come from D_n are going to be different to those that come from P_n, so we avoid repetition, unless it is a linear order. But we know how to detect those very easily, since we know them all, they are the $\llbracket n \rrbracket$!

3. Tabulation of POSETS

The Merriam-Webster Dictionary defines

Tabulate: *to count, record, or list systematically.*

In mathematics the process of tabulation is important and we have seen one of the end products in the case of knots: the knots tables, ordered by *intersection number*. It looks like this:



It is important to emphasize that the tables or lists produced are organized in some way (in the case of knots, by intersection number) which allows us to locate the objects being studied within the table.

The other relevant aspect is that we have information associated to the elements of the table, which implies that whenever we find an object that we can then locate in our table, we can deduce its properties which we already have computed.

Our objectve is to create a similar, nice looking, table for certain POSETS!

(a) In the previous exercise you have computed several Hasse Diagrams of several POSETS of a particular kind.

Let us give a definition now, whose idea we introduced in class, but we did not called it like this:

> A **finite** POSET $(P, \leq)$ is a **interval POSET**, or just interval, if there exists one element a such that $a \leq x$ for all x in P and one element b such that $x \leq b$ for all x in P. That is, there is a smallest and a biggest element. We call these elements the left endpoint and right endpoint

Explain why $[\![n]\!]$, D_n, P_n are all interval POSETS

(b) Tabulate all intervals of 6 or less elements, up to isomorphism, by creating a table of their Hasse Diagram organized by number of elements. You should explain how did you find these intervals, why the list is complete and why it is nonredundant.

Some suggestions to achieve this: (1) Search in the families that you have, or invent a new family if you can come up with one. (2) Try to construct new intervals from older ones with fewer elements.

> A priori you can look in google for this table, but don't do it. The point is that you actually experience the process of finding, tabulating and knowing you are not missing anything, so that you appreciate how tricky it can be. I will show the table in the next class.

(c) Which of the elements of the previous table **do not** appear in the families $[\![n]\!]$, D_n, P_n?