University of Toronto

Classification in Mathematics

Due: May 30, 2020

Try to solve the most you can and explain the solution to the best of your abilities. Do not just put computations, be mindful of presentations and of the quality of your writing.
Please upload the solutions with the right orientation, since reading rotated images is very tiring and sometimes complicated.
Make sure your problems are in order.
These problems might be harder for some of you than others, I do encourage you to talk among yourselves if that's possible and to ask the TA or me for suggestions and help. If you get frustrated, GOOD, is part of progress. Don't let it dominate you, look for help, think, take your time. Math is worth the challenge.
For this homework we do expect everybody to submit the homework on time and with complete solutions for the 5 problems.

1 Concepts

We have introduced during the lecture the following concept:

A **group** is a set G with an operation $\circ$ such that

- for all $a, b \in G$ we have $a \circ b \in G$,
- there exists an element $e \in G$ (which we called 1 sometimes during lecture) such that $e \circ g = g \circ e = g$ for all $g \in G$,
- for every $g \in G$ there exists $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$,
- the operation is associative: $(a \circ b) \circ c = a \circ (b \circ c)$ for all a, b, c in G.

We will only focus on finite groups in this course but there are infinite groups and they are very natural and important.

We saw some examples of groups:

The **symmetric group** S_n on n elements is the set of all permutations of a set of n elements, which we usually denote by $1, 2, ..., n$. The operation $\circ$ is the composition of permutations, that is, making one permutation and then the other one.

The other example we have seen is:

The **dihedral group** D_n are the symmetries of a regular $n-$gon. We have seen the examples of an equilateral triangle and the regular pentagon in class.

Finally, we also introduced the idea of the "clock arithmetic" group:

The **cyclic group** of order n (we didn't call it like this during the lecture) is the group whose operation is obtained by doing clock arithmetic in a clock with n hours
We saw the example of 6 hours in class.

A concept we mentioned during the lecture while doing examples is:

Two groups G and H are **isomorphic** if there is a one-to-one onto function $f : G \to H$ such that the product is preserved:

$$f(a \circ_G b) = f(a) \circ_H f(b).$$

In more concrete terms, if their multiplication tables become the same one after relabeling the elements and possible reordering the columns and the rows of the table.

2 Problems

1. Tabulation
   (a) Find up to isomorphism all finite multiplication tables up to order 6. That is, find all groups, up to isomorphism with no more than 6 elements.
   (b) Let p be a prime number. Can you find a reason why, up to isomorphism, there is only one group of p elements? Produce an example of such a group.
2. The symmetric group
   (a) Justify that the number of elements of the symmetric group S_n is $n!$
   (b) Suppose σ is a permutation in S_4 (i.e. it permutes the elements $1, 2, 3, 4$). Let us consider four different variables x_1, x_2, x_3, x_4 and a polynomial $P(x_1, x_2, x_3, x_4)$ in those four variables.

   Examples of such polynomials are: $x_1x_2x_3x_4$ or x_3^2. Notice not all variables need to appear in the polynomial.

   Given our permutation, we will construct a new polynomial:

$$(\sigma \cdot P)(x_1, x_2, x_3, x_4) = P(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)})$$

   For example, for the permutation $\sigma(1) = 1, \sigma(2) = 3, \sigma(3) = 4, \sigma(4) = 2$ and the polynomial $x_1x_2x_4$ we get

$$\sigma \cdot x_1x_2x_4 = x_1x_4x_3.$$

   For the polynomial $P(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$ make a table that finds $\sigma \cdot P$ as σ varies in S_4 and write down which elements satisfy $\sigma \cdot P = P$ and how many different polynomials you get while doing this.
   (c) Prove that the set of elements of S_4 that satisfied, in the previous part, $\sigma \cdot P = P$ is a group. Which multiplication table, of those you found in problem 1 corresponds to it?
   (d) (Optional) A polynomial in 4 variables is symmetric if $\sigma \cdot P = P$ for all σ in S_4. The four standard symmetric polynomials are:

$$\begin{aligned}
\sigma_0 &= x_1x_2x_3x_4, \\
\sigma_1 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \\
\sigma_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\
\sigma_3 &= x_1 + x_2 + x_3 + x_4
\end{aligned}$$

Prove that if P is a symmetric polynomial then $P = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, for some numbers a_0, a_1, a_2, a_3.

Newton introduced these polynomials when he gave his proof of the binomial theorem and Gauss proved the general version (i.e. for n variables) of this result.

3. Dihedral Groups and related
   (a) Find the number of elements of the Dihedral group D_n.
   (b) Draw a point in the $x-y$ plane for each one of the following coordinates $(1,0), (0,1), (-1,0), (0,-1)$ and $(1,1), (1,-1), (-1,1), (-1,-1)$ Show your graph.
   (c) Find all symmetries of this set of points, which preserve distances between points, in the same way we found all symmetries of a regular n-gon.
   (d) Of all those transformations that you found in the previous part, is there any point which has more than two symmetries fixing it?
   (e) Given any two points of the previous part, what requirement must they satisfy to make sure there is a symmetry taking one into the other?
4. Cyclic groups
   (a) A finite cyclic group, in general, is one whose elements are $1, g, g^2, ..., g^{n-1}$. That is, all the elements are just composing a single element with itself.

       Prove that all cyclic groups are indeed the "clock" group for some n.
   (b) Not all groups are cyclic groups, but we can build cyclic groups inside another given group by just taking powers of a single element.

       Prove that the rotations form a cyclic group inside the dihedral group.
   (c) Justify that all cyclic groups are commutative. That is: $a \circ b = b \circ a$.
   (d) Find a commutative group that is not cyclic among those you tabulated in the first problem.
5. More Tabulation

   Now that you have gathered some ideas of groups and their properties, complete your tabulated table to include all finite groups, up to isomorphism, up to 9 elements.