University of Toronto

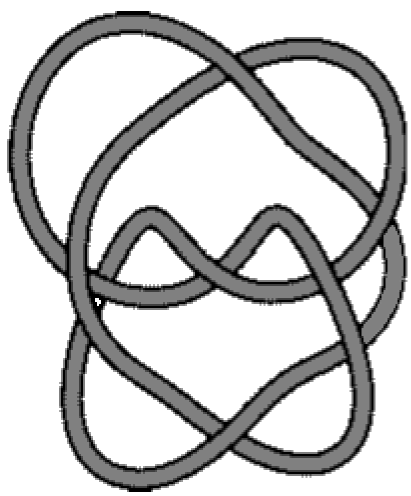
Math Academy 2021

August 13, 2024

Try to solve the most you can and explain the solution to the best of your abilities. Do not just put computations, be mindful of your arguments. Discuss with the other members of the academy and learn together!
These problems might be harder for some of you than others, I do encourage you to talk among yourselves if that's possible and to ask me for suggestions and help if you need. If you get frustrated, GOOD, is part of progress. Don't let it dominate you, look for help, think, take your time. Math is worth the challenge.

1 Problems

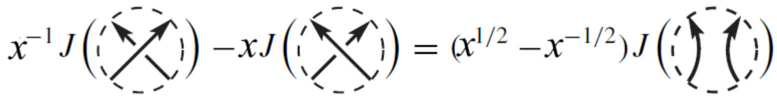
1. Compute the Conway Polynomial and the Jones Polynomial for the following knot, which is 9_{46}.



   Notice that this is not an alternating diagram.

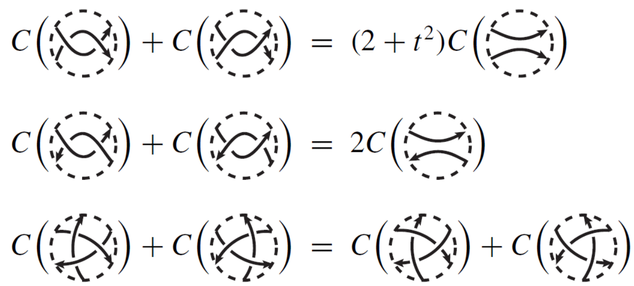
2. We have seen that the Conway Polynomial and the Jones Polynomial both are defined by an initial condition having value 1 at the unknot and a particular Skein Relationship. For the Conway Polynomial it is

   while for the Jones Polynomial it is

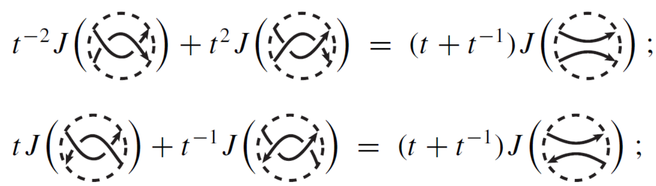


Only knowing this several properties can be obtained.

(a) Prove that the Conway Polynomial satisfies



(b) Prove that the Jones Polynomial satisfies



(c) Find the analogous of the third equation of the Conway Polynomial but for the Jones Polynomial.

3. (a) Prove that, for knots (and not links!!) both the Jones and Conway Polynomial are invariant under change of orientation.

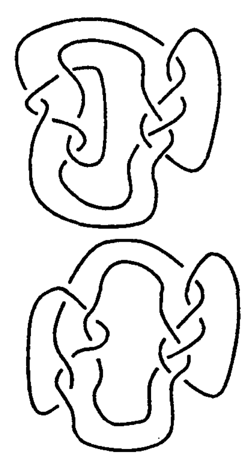
   (b) For a knot K, let K^* be its mirror reflection (i.e. change upper crossings for undercrossings and viceversa). Prove that

$$J_K(t) = J_{K^*}(t^{-1}).$$

   J_K and J_{K^*} stand for the Jones Polynomial of K and K^* respectively.

   (c) What happens under mirror reflection for the Conway Polynomial?

4. We have seen that the Jones Polynomial does detect the different trefoils (i.e. left from right trefoil.) The following knots are called the Conway Knot and the Kinoshita-Terasaka knot.



(a) Prove that their Jones Polynomial are equal.

(b) Compute their Conway Polynomials and verify with this that these knots are actually different.

The way in which these knots differ one from the other is called **mutation**. This term was introduced by Conway.

5. (a) Prove that the initial condition and the Skein relationship for the Conway Polynomial define it uniquely. That is, there is a unique way to associate a Polynomial to each oriented link diagram such that these relationships are satisfied.

   (b) Prove that the Conway Polynomial is invariant under Reidemeister Movements.

2 Appendix: Knot Tables