University of Toronto
Math Academy 2020

Exercise Sheet 1

Please try these by yourself and also look help of your fellow students and me. Make your best. We will be discussing several things of these problems in the office hours, in class, in the chat you can discuss as well. Try to learn a lot, write down the solutions and make sure you dedicate time to trying these questions. They are not straight forward.

1. For the following type of sequences find an Euclid Type of proof to show there exist infinitely many primes in such an arithmetic sequence:

   - $4k + 1$
   - $4k - 1$
   - $3k + 1$
   - $3k - 1$
   - $5k + 2$

   Online Research: What is the condition needed for an Euclid Type of Proof to exist for the sequence $ak + b$? Can you conjecture it by yourself?

2. For residues modulo 7 (i.e. when dividing by 7) do the following: call $\pi_{7,i}(x)$ the number of primes that leave remainder i when divided by 7 and smaller or equal to x, for $i = 1, 2, 3, 4, 5, 6$. Using some sieve methods first by hand, and then by the aid of a computer (and your fellow students if you do not know how to code), compute $\pi_{7,i}(x)$ for each i and for values of x of to 100,000.

   Are these number distributed in the same way, or is there some subsequences more popular than others? What happens if you change the 7 by another number, for example by 9?

   Online Research: Try to play with these by yourself first, but is there some sort of prime number theorem for $\pi_{7,i}(x)$?

3. We have seen that primes can be found in arithmetic sequences, but that doesnt mean that they themselves form an arithmetic sequence. Find five prime numbers that form an arithmetic progression. Can you find 7?

   Using a computer code (or help from a fellow student) find all arithmetic progressions made only of prime numbers smaller than 100,000. Which is the largest one? Which length repeats more often?

   Online Research: What is the Green-Tao Theorem? After who is these theorem named?

4. In here we have studied the infiniteness of primes in arithmetic progressions $ak + b$, but what about other families? Which primes do you conjecture appear in the following families:

   - $x^2 + y^2$ as x, y vary in the integers.
   - $x^2 + 2y^2$ as x, y vary in the integers.
   - $x^2 + y^2 + z^2$ as x, y, z vary in the integers.

- $x^2 + y^2 + z^2 + w^2$ as x, y, z, w vary in the integers.

Research online: What happens with the family $x^2 + y^4$. It has been proven there are infinitely many primes in this sequence or not? If so, who proved it?

5. Modify the proof for the infinite number of primes of the form $4k+1$ that we did in class for it to apply for $4k-1$. Can you modify it for it to work for $3k+1$? What about $3k-1$?

   Research Online: What is a Dirichlet Character? Who is Dirichlet? Why was this theorem of arithmetic progressions important historically as a stepping stone for another theorem called The Law of Quadratic Reciprocity.

6. (Preparing ourselves for the future!) What is ergodic theory? What do we mean by that? What is topology? What is statistics and expected value?

   Do not give "rigurous math" answers to this, read and discuss with your fellow students to get an intuitive idea of what we really mean as an idea, even if we do not get the details yet.

7. Experiment: Online look for a dice of a 100 sides and let it play N times. Say $N = 100$ and then $N = 1000$. Record how many times it is a prime number? What do you notice? What is the relationship of this to the prime number theorem?