University of Toronto
Math Academy 2020

Exercise Sheet 2

1. The vertices of a graph G can be coloured with two colours without an edge having its two ends of the same colour if and only if it contains not odd cycles.

   An odd cycle is a sequence of an odd number of edges edges that begins and ends in the same place, and where two consecutive edges are different but share a common endpoint.

   Research Online: What is the chromatic number of a graph and what is a chromatic polynomial. How does this relate to the 4 colour theorem?

2. Solve the following questions related to probability:

   (a) A traditional fair dice is thrown twice. What is the probability that

   1. a six turns up exactly one.
   2. both numbers are odd.
   3. the sum of the scores is 4.
   4. the sum of the scores is divisible by three.

   (b) You are travelling on a train with your best friend. Neither of you has a valid ticket and the inspector has caught you both (oh no!). He is authorized to administer a special punishment for this offense (oh oh!). He holds a box containing nine apparently identical chocolates, but three of these are contaminated with a deadly poison. He makes each of you, in turn, chose and immediately eat a single chocolate.

   1. What is the probability you both survive?
   2. What is the probability you both die?
   3. If you choose first, what is the probability you survive but your best friend dies?

   (c) We create a random graph, similar to what we did in the course as follows: On a set of n vertices, for each possible edge we draw it with probability p and do not draw it with probability $1-p$. Let X be the number of isolated vertices (i.e. the number of vertices not joined to some other vertex.). What is the expected value for X?

3. Recall we have defined expected value of some experiment X as long as X takes a discrete set of real values (for example the integers, the positive integers, etc.). Prove the following inequalities:

   (a) Let X be an experiment with expected value $E(X)$. Suppose that $E(X) \leq t$, where t is some real number. Prove then that $\mathbb{P}(X \leq t) > 0$. (This is called the First Moment Principle).

   (b) Prove that if X only takes positive variables, then $\mathbb{P}(X \leq t) \leq E(X)/t$.

   (c) Prove that if X and Y are two experiments, then $E(X+Y) = EX + EY$. This is called linearity of expectation.

4. In this problem we will prove the following result: *For any $k \geq 1$ there exists triangle free graphs with chromatic number greater than k.* Let G be a random graph in n vertices and in which an edge is drawn with probability $p = n^{-2/3}$.

   (a) A subset of vertices of the graph is stable if there are no edges between them. For $n = 5$ and $k = 3$, run the experiment several times, just as we did in class (if you code the experiment do it some 100 times) and compute how many stable sets of vertices there are of size 2.

(b) For a fixed n and k let I be the question: how many stable subets of size $\lceil n/2k \rceil$ (this means, the nearest bigger integer to $n/2k$). Furthermore, for each set S of size $\lceil n/2k \rceil$ let I_S be 1 if S is stable and 0 if it is not. Prove that

$$E(I_S) = (1-p)^{\binom{\lceil n/2k \rceil}{2}}$$

(c) Use linearity of expectation to write I in terms of the I_S, as S varies in the stable sets of size $\lceil n/2k \rceil$, and to compute

$$E(I) < \binom{n}{\lceil n/2k \rceil}(1-p)^{\binom{n/2k}{2}}$$

(d) Prove (or explore graphically) why for positive real numbers x we have

$$1 - x \leq e^{-x}.$$

(e) Prove that for $n \geq 2^{12}k^6$ we have $E(I) \leq 1/2$. Use Markov inequality to conclude then

$$\mathbb{P}(I > 0) \leq 1/2.$$

(f) Let T be the question: how many triangles are there in G? For the experiment you did in part (a), now compute this value of T and take the actual average of the numbers you got.

(g) Prove that $E(T) = \binom{n}{3}p^3$. Is this consistent with your approximation in the previous part? What is the theorem that tells you this had to be the case if you repeated the experiment enough times?

(h) Conclude from the previous part that $E(T) < n/3$, and hence that

$$\mathbb{P}(T \geq n/2) < 1/3$$

if n is large enough.

(i) Justify that $\mathbb{P}(I \geq 1) + \mathbb{P}(T \geq n/2) < 1$ and conclude that the probability that $I = 0$ and $T < n/2$ is positive, if n is large enough.

(j) Conclude from the previous part that there exists a graph G, if n is large enough, without stable sets of size $\lceil n/2k \rceil$ and at most $n/2$ triangles. (Here is where the probabilistic method happened!).

(k) For the graph you proved exists in the previous part, chose a set of the vertices of G, with at most $n/2$ elements, and with at least one vertex of each one of the existing triangles in G. Explain why this is possible.

(l) Delete those vertices and the edges joined to them, and call G' the resulting graph. Justify that the number of vertices of G' is at least $n/2$ and that G' has no triangles.

(m) Prove that this graph G' has chromatic number bigger than k.

(n) Write down this proof in your own words so that you realize how the probabilistic method, via the computations of expectations and Markov inequality, actually proved the existence of such a graph without actually being completely constructive.

(o) For a given n and k, could you construct a program code that finds such a graph, out of the ideas in this proof?.

Research Online: This theorem was proved before the probabilistic method was introduced. The proof shown here is due to Erdos. Who proved this result before, and why is the probabilistic method such an improvement to what was known before?

5. Modify the proof of the previous problem 4 to prove the result: *For any $k \geq 1$ and $g \geq 3$ there exists graphs with chromatic number greater than k and with no cycles of length at most g.*

   It would be very nice if someone presents to the class the proof of this result carefully, and with some examples, at some point of next week so that we can practice the power of the probabilistic method.